Second "Mid"-Term for M365C

- This is an exam. You may consult your notes and textbooks but MUST NOT consult other people.
- Please write neatly on only 1 side of the sheet of paper.
- Please use a separate sheet for each question.
- All problems are of equal value though not necessarily equal difficulty.

Due Monday December 6th by 5 pm

(1) Let $f : [0, +\infty) \to \mathbb{R}$ be Riemann integrable on $[0, b]$ for every $b > 0$. We say the improper integral $\int_0^\infty f(x)dx$ converges if

$$\lim_{b \to +\infty} \int_0^B f(x)dx$$

exists and is finite. In this case we write

$$\int_0^\infty f(x)dx = \lim_{b \to +\infty} \int_0^\infty f(x)dx.$$

(a) Find $f, g : [0, +\infty) \to \mathbb{R}$ such that both $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ coverge but such that $\int_0^\infty f(x)g(x)dx$ does not converge.

(b) Show that if $\int_0^\infty f(x)^2 dx$ and $\int_0^\infty g(x)^2 dx$ converge then so does $\int_0^\infty f(x)g(x)dx$.

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and suppose there exists L such that

$$|f(x_1, t) - f(x_2, t)| \leq L|x_1 - x_2|$$

Fix $(x_0, t_0) \in \mathbb{R}^2$ and define

$$I(\varphi)(t) = \int_{t_0}^t f(\varphi(s), s)ds + x_0$$

Show that if $\delta > 0$ is chosen small enough then I is a contraction mapping on $C\big([t_0 - \delta, t_0 + \delta], \mathbb{R}\big)$, that is there exists $\lambda < 1$ such that

$$\|I(\varphi_1) - I(\varphi_2)\|_0 \leq \lambda \|\varphi_1 - \varphi_2\|_0$$

Thus conclude I has a unique fixed point and hence show that there exists a unique solution to

$$\frac{dx}{dt} = f(x, t), \qquad x(t_0) = x_0$$

for $t \in [t_0 - \delta, t_0 + \delta]$.

Remark: This is called Picard's Theorem. In reality it is only necessary that the function f be continuous and satisfy such a Lipshitz condition locally though then the proof is slightly more tricky. If we only assume continuity then you can still show that a solution exists though it need not be unique. This is called Peano's Theorem and uses the Ascoli-Arzela theorem.

(3) Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)(x)}$, the n-th deivative of f at x, is continuous on $[a, b]$ and differentiable on (a, b). Let $x_0 \in (a, b)$ and define the n-th order Taylor polynomial about x_0 by

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x - x_0)^i.$$

Show that for each $x \in [a, b]$ there exists ξ between x and x_0 such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

This is known as the Lagrange form of the remainder.

[Hint: Consider $g(t) = f(t) - P_n(t) - C(t - x_0)^{n+1}$ where C is chosen so that $g(x) = 0$. Then $g(x) = g(x_0) = 0$ and we can begin iteratively applying Rolles Lemma.]

(4) Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n+1)}(x)$, the $n + 1$-th deivative of f at x, is defined and continuous for all $x \in [a, b]$. Let $x_0 \in (a, b)$ and define the n-th order Taylor polynomial about x_0 by

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x - x_0)^i.$$

Show that for each $a \leq x \leq n$

$$f(x) = P_n(x) + \int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x - t)^n dt$$

This is often known as the integral form of the remainder.

[Hint: Proceed by integration by parts.]