Homework 2 for M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Monday September 13th at 9 am

(1) Show that a countable union of countable sets is countable. I.e. If I is a countable index set and for each $i \in I$ A_i is a countable set then

$$\bigcap_{i \in I} A_i$$

is countable.

(2) Let A and B be sets. Define

$$X = \{(C, D, f) : C \subset A, D \subset B, f : C \to D \text{ a bijection}\}$$

We can put a partial order on X as follows $(C, D, f) \leq (C', D', f')$ if $C \subset C'$, $D \subset D'$, and f' extends f that is for all $x \in C$ $f'(x) = f(x)$.

(a) Show that every chain in X has an upper bound in X.

(b) By Zorn's Lemma X contains a maximal element. Show that either there is a subset $C \subset A$ such that the maximal element is (C, B, f) or there is a subset $D \subset B$ such that the maximal element is (A, D, f).

(3) Show that $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$ with the distance

$$d\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

is a metric space.

(4) Let (X, d) be a metric space. Let $C \subset X$ be a closed subset. Let (x_n) be a sequence with $x_n \in C$ for all n. Show that if $\lim x_n = x$ then $x \in C$.