

HOMWORK 3 FOR M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE MONDAY SEPTEMBER 20TH AT 9 AM

- (1) Prove that:
 - (a) If $\{U_\alpha : \alpha \in I\}$ is a set of open sets then $\bigcup_{\alpha \in I} U_\alpha$ is an open set.
 - (b) If $\{U_i\}_{i=1}^n$ is a finite set of open sets then $\bigcap_{i=1}^n U_i$ is an open set.
 - (c) Give an example to show that a countable intersection of open sets need not be open.
- (2) Show that $l_\infty := \{(x_n) : \exists M \forall n |x_n| < M\}$ with the metric $d_\infty((x_n), (y_n)) := \sup_n |x_n - y_n|$ is a complete metric space.
- (3) Let (X, d) be a complete metric space and suppose $f : X \rightarrow X$ is a function with the property that there exists $\lambda < 1$ such that

$$d(f(x), f(y)) < \lambda d(x, y).$$

Prove that there exists a unique point $x \in X$ such that $f(x) = x$. [Hint for Existence of a Fixed Point: Pick $x_0 \in X$ and let $x_{n+1} = f(x_n)$. Show the sequence is Cauchy and hence converges and that the limit must be a fixed point.]

- (4) A metric space (X, d) is totally bounded if for every $\epsilon > 0$ there exists x_1, \dots, x_n such that

$$X = \bigcup_{i=1}^n B(x_i, \epsilon).$$

Prove that a totally bounded metric space is (sequentially) compact if it is totally bounded.

Hint: Mimic the proof of the Bolzano-Weierstrass result. Prove that a subset of a totally bounded space is totally bounded.

Remark: In fact this is an if and only if result.