Homework 4 for M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Monday September 27th at 9 am

(1) A subset $A \subset (X, d)$ is called an ϵ-net if

$$X = \bigcup_{x \in A} B(x, \epsilon)$$

A metric space (X, d) is called totally bounded if for every $\epsilon > 0$ there is a finite ϵ-net.

Prove that a complete metric space is (sequentially) compact if it is totally bounded.

Hint: Mimic the proof of the Bolanzo-Weierstrass result. Prove that a subset of a totally bounded space is totally bounded.

Remark: In fact this is an if and only if result. We will prove the other direction in class.

(2) Give an example of a sequence of nested closed sets (A_n) in a $\mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

(3) Show that if (X, d) is a sequentially compact metric space then for any sequence of nested closed sets (A_n)

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

(4) Show that the closed ball $\overline{B}(0, 1) \subset l_\infty$ is not sequentially compact.