Homework 5 for M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Monday October 18th at 9 am

(1) Show that $f : (X, d_X) \to (Y, d_Y)$ is continuous on X if and only if for every open $U \subset Y$ the pre-image
$$f^{-1}(U) = \{x \in X : f(x) \in U\}$$
is open.

(2) Prove that if $f : (X, d_X) \to (Y, d_Y)$ is continuous and (X, d_x) is compact then $f(X)$ is a compact subset of Y (that is every sequence in $f(X)$ contains a convergent subsequence with limit in $f(X)$).

(3) Show that
$$C\big([0,1], \mathbb{R}\big) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$$
with the norm
$$\|f\| = \max_{x \in [0,1]} |f(x)|$$
is a normed vector space.

(4) Show that the map $I : C\big([0,1], \mathbb{R}\big) \to \mathbb{R}$ defined by
$$I(f) = \int_0^1 f(x)dx$$
is continuous. You may use all the basic properties of the integral from your calculus courses.

(5) Show that the map $\delta_0 : C\big([0,1], \mathbb{R}\big) \to \mathbb{R}$ defined by
$$\delta_0(f) = f(0)$$
is continuous.