

## HOMWORK 6 FOR M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE MONDAY OCTOBER 25TH AT 9 AM

- (1) Prove that if  $(X, d)$  is path connected then  $(X, d)$  is connected.
- (2) Prove that if  $(X, d_X)$  is connected and  $f : (X, d_X) \rightarrow (Y, d_Y)$  is continuous then  $f(X)$  is connected.
- (3) Given a countable collection of metric space  $(X_i, d_i)$  we can define a metric on the Cartesian product  $X = \prod_{i=1}^{\infty} X_i$  by

$$d_b((x_i), (y_i)) = \sup_i \hat{d}_i(x_i, y_i)$$

where  $\hat{d}_i(x_i, y_i) = \min\{1, d_i(x_i, y_i)\}$ . Show that the sequence  $((x_i^{(n)}))$  in  $\mathbb{R}^{\infty} = \prod_{i=1}^{\infty} \mathbb{R}$  defined by

$$x_i^{(n)} = \frac{i}{n}$$

has the property that  $(x_i^{(n)}) \subset \mathbb{R}$  converges for every  $i$  but  $((x_i^{(n)}))$  does not converge in  $(\infty, d_b)$ . Contrast this with the usual product metric.

- (4) Prove that  $f : (X, d_X) \rightarrow (Y, d_Y)$  is continuous if and only if for every  $C \subset Y$  closed  $f^{-1}(C) \subset X$  is closed.
- (5) Prove that if  $f : (X, d_X) \rightarrow (Y, d_Y)$  is a continuous function and  $(X, d_X)$  is compact then  $f$  is a closed mapping (that is if  $C \subset X$  is closed then  $f(C) \subset Y$  is closed). Hence show that if  $f : (X, d_X) \rightarrow (Y, d_Y)$  is a continuous bijection and  $(X, d_X)$  is compact then  $f$  is a homeomorphism (Hint: Use the previous question).