HOMEWORK 8 FOR M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE MONDAY NOVEMBER 8TH AT 9 AM

(1) Each of the following functions is defined on $(0,1]$. Extend each to a function $f:[0,1]\to\mathbb{R}$ by setting $f(0)=0$. For each function decide whether it is Riemann integrable on $[0,1]$ and give a proof.
  (a) $\sin^2(\frac{1}{x})$
  (b) $\frac{1}{x}\sin(\frac{1}{x})$.
  (c) $\ln(x)$.
  (d) $\frac{\sin(x)}{x}$. Hint: Show $\sin x < x$ for $0 < x \le \pi$.

(2) Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \end{cases}$$

Let $f:[a,b]\to\mathbb{R}$, f continuous, with $a < 0 < b$. Compute $\int_a^b f dg$.

(3) Given functions $f, g : [a,b]\to\mathbb{R}$ we define the Riemann-Stieltjes sum of f with respect to partition $\pi = \{x_i\}_{i=0}^n$ and sample $\sigma = \{\xi_i\}_{i=1}^n$ to be

$$S_g(f,\pi,\sigma) = \sum_{i=1}^n f(\xi_i)(g(x_i) - g(x_{i-1}))$$

and then

$$\overline{S}_g(f,\pi) = \sup\{S_g(f,\pi,\sigma) : \sigma \text{ associated to } \pi\}$$

Show that if g is not monotone then it is not necessarily true that

$$\overline{S}_g(f,\pi') \le \overline{S}_g(f,\pi)$$

for $\pi \le \pi'$. Hint : Consider the effect of intervals $[x_{i-1}, x_i]$ with $g(x_i) - g(x_{i-1}) < 0$.