Math 343K Exam 2 Solutions

Question 1

1. Let H_i, for i in an index set I, be subgroups of $< G, * >$. Define

$$H = \bigcap_{i \in I} H_i.$$

Prove that H is a subgroup of G.

Solution:

We must show that H contains the identity, is closed under taking inverses, and is closed under the operation $*$.

   (a) Since each H_i is a subgroup we have $e \in H_i$ for all $i \in I$, hence by definition of the intersection $e \in H$.

   (b) Suppose $h \in H$ then by definition of the intersection $h \in H_i$ for all i. Since H_i is a subgroup it is closed under taking inverses, thus $h^{-1} \in H_i$ for all i. Thus by definition of the intersection $h^{-1} \in H$.

   (c) Suppose $h, k \in H$ then by definition of the intersection $h, k \in H_i$ for all $i \in I$. Since each H_i is a subgroup it is closed under the operation, thus $h * k \in H_i$ for all $i \in I$. Thus by the definition of intersection $h * k \in H$.

2. Find, up to isomorphism, all abelian groups of order 48.

Solution:

There are 5 such groups.

   (a) $\mathbb{Z}_{48} \simeq \mathbb{Z}_{16} \times \mathbb{Z}_3$

   (b) $\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3$

   (c) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_4 \times \mathbb{Z}_3$

   (d) $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3$

   (e) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

3. Show that $\mathbb{Z}_{12} \times \mathbb{Z}_3$ is not isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_6$.

Solution:

The element $(1, 0) \in \mathbb{Z}_{12} \times \mathbb{Z}_3$ has order 12. However for all $(a, b) \in \mathbb{Z}_6 \times \mathbb{Z}_6$ we have

$$6(a, b) = (6a \bmod 6, 6b \bmod 6) = (0, 0)$$

so that (a, b) has order no more than 6. Since there are no elements of order 12 in $\mathbb{Z}_6 \times \mathbb{Z}_6$ it cannot be isomorphic to $\mathbb{Z}_{12} \times \mathbb{Z}_3$.

Question 2

1. Define what it means for a permutation $\sigma \in S_n$ to be an even permutation.

   Solution:

   A permutation $\sigma \in S_n$ is called even if it can be written as the product of an even number of transpositions. We showed in class that this definition is well defined.

2. Write the following permutation as a product of disjoint cycles and say whether it is an even or an odd permutation. *Justify your answer.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 1 & 2 & 4 & 7 & 6 \end{pmatrix}$$

   Solution:

   $\sigma = (13)(254)(67)$. The 3-cycle (254) can be written as $(254) = (24)(25)$. Thus $\sigma = (13)(24)(25)(67)$ and is even. Equivalently we can just note that σ is the product of 2 odd permutations and 1 even permutation and that 2 odd numbers plus and even number is even.

3. Prove that a cycle σ of length m is

   (a) an even permutation if m is odd.

   (b) an odd permutation if m is even.

   You need only write σ in the required form.

   Solution:

   If $\sigma = (a_1 \cdots a_m)$ then it can be written

$$\sigma = (a_m a_1)(a_{m-1} a_1) \cdots (a_2 a_1).$$

   There are $m - 1$ transpositions so the cycles is even if m is odd and the cycle is odd if m is even.

4. State Cayley's Theorem.

   Solution:

   Every group G is isomorphic to a group of permutations (in particular to a subgroup of S_G).

Question 3

1. Let H be a subgroup of $< G, * >$. Define aH, the left coset of H containing a.
   Solution: The left coset of H containing a is given by
   $$aH := \{a * h : h \in H\}$$

2. Prove that if $aH \cap bH \neq \varnothing$ then $aH = bH$. *Hint: By symmetry it suffices to show that $aH \subseteq bH$.*

   Solution:

   Assume $aH \cap bH \neq \varnothing$ and let $c \in aH \cap bH$. By definition of the cosets $c = ah_1$ and $c = bh_2$ for some $h_1, h_2 \in H$. Thus $ah_1 = bh_2$ and we can write $a = bh_2h_1^{-1}$. Now let $d \in aH$ be arbitrary. By definition of the coset $d = ah_3$ for some $h_3 \in H$. Thus $d = bh_2h_1^{-1})h_3$ by substitution. However since H is a subgroup we see $h_2h_1^{-1})h_3 \in H$ and thus we have $d \in bH$. Since $d \in aH$ was arbitrary we have $aH \subseteq bH$. Since the everything is symmetric in a and b the same argument gives $bH \subseteq aH$ and thus $aH = bH$.

3. Find the left cosets of $H = \{0, 3\}$ in the group $< \mathbb{Z}_6, +_6 >$.

   Solution:
   $$0 + H = 3 + H = \{0, 3\}$$
   $$1 + H = 4 + H = \{1, 4\}$$
   $$2 + H = 5 + H = \{2, 5\}$$

4. State Lagrange's Theorem.

   Solution:

   If H is a subgroup of G then the order of H must divide the order of G.

5. Prove that every group G of prime order is cyclic.

   Solution:

   If G is the trivial group then it is cyclic. If G is not trivial then choose $g \in G$ with $g \neq e$. The cyclic group generated by g, $\langle g \rangle$, must have at least 2 elements. However the order of $\langle g \rangle$ must divide the order of G. Since the order of G is a prime the only divisors are $|G|$ and 1. Since $|\langle g \rangle| \neq 1$ we must have $|\langle g \rangle| = |G|$ and hence $G = \langle g \rangle$. Thus G is cyclic.

Question 4

1. Define *Frieze group.*

   Solution:

   A Frieze group is an **infinite discrete** subgroup of the group of plane isometries that leaves a **line invariant**.

2. Describe the different isometries that are allowed to appear in a Frieze group.

   Solution:

   There are 5 distinct types of non-identity transformations that appear.

   (a) Translations along the invariant line.
   (b) Glide reflections along the invariant line.
   (c) Half turns about a point on the invariant line.
   (d) Reflections in the invariant line.
   (e) Reflections in an axis orthogonal to the invariant line.

3. Give the *crystallographic restriction* and explain when it applies.

   Solution:

   The crystallographic restriction says that only rotations of order 2, 3, 4, or 6 can appear in an infinite discrete subgroup of the group of plane isometries (or the group of isometries of Euclidean 3-space).

4. There is one type of Frieze group which is abelian but not cyclic. Describe its generators and draw a Frieze pattern which has this group as it largest group of symmetries.

   Solution:

   Since the group is not cyclic it must have more than one generator. Every Frieze group has either a translation or glide reflection as one of its generators. If the group is to be commutative then the generators must commute. The only transformation that commutes with a translation or with a glide reflection is the reflection in the invariant line. The half turn and the reflection in an orthogonal axis both reverse the direction of the translation or glide reflection. Thus the group is the one generated by a translation and a reflection in the invariant line (or equivalently the group generated by a glide reflection and a reflection in the invariant line). This is the group labeled F_6.