Homework 2 for M343K

- Please label your homework clearly with your name.
- Homework must be neatly written and stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.
- All solutions must take the form of complete sentences.

Due Tuesday September 20th

(1) Define the relation $\simeq$ on the set of all binary structures by $< S, * > \simeq < T, \star >$ if there exists $\varphi : S \to T$ a bijection such that for all $a, b \in S$

$$\varphi(a) \star \varphi(b) = \varphi(a * b).$$

Show that this relation is an equivalence relation, i.e. it is reflexive, symmetric, and transitive.

(2) Show that "the operation is associative" is a structural property, i.e if $< S, * >$ is a binary structure with $*$ associative and $< T, \star >$ is isomorphic to $< S, * >$ then $\star$ is associative.

(3) Let S be a set with 4 elements. How many ways are there of defining an operation $*$ on S such that $< S, * >$ is a group ?

(4) Show that the number of solutions of the equation $x^n = e$ for each $n \in \mathbb{Z}^+$ is a structural property.

Remark: More properly the cardinality of the set of solutions is a structural property. Any isomorphism of two groups must establish a bijection between the sets of solutions. For an infinite group there need not be any non-trivial solutions for $n \geq 2$ but for groups with N elements every element satisfies $x^N = e$ and non-trivial solutions of $x^n = e$ for $2 \leq n < N$ may exist too.

(5) Show that $< S^1, \times >$ is not isomorphic to either $< \mathbb{R}, + >$ or $< \mathbb{R}^\times, \times >$.

Hint: Consider solutions of the equations $x^n = e$ in each of the groups.

(6) Suppose $< G, * >$ is agroup with an even number of elements. Show there must be an element $a \in G$ with $a \neq e$ such that $a * a = e$.

Hint: Consider pairing elements with their inverses.