Homework 6 for M343K

- Please label your homework clearly with your name.
- Homework must be neatly written and stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.
- All solutions must take the form of complete sentences.

Due Tuesday October 27th

(1) Let A be an infinite set. We say $\sigma \in S_A$ moves $a \in A$ if $\sigma(a) \neq a$. Let H be the set of $\sigma \in S_A$ that move only finitely many elements. Show that H is a subgroup of S_A.

(2) Suppose H and K are subgroups of $< G, * >$ such that $K \leq H \leq G$. If both $(G : H)$ and $(H : K)$ are finite then $(G : K)$ is finite and $(G : K) = (G : H)(H : K)$.

(3) Show that if $< G, * >$ has order n then $a^n = e$ for all $a \in G$.

(4) Let H be a subgroup of $< G, * >$. Show that there are the same number of left and right cosets by finding a bijection between the set of left cosets and the set of right cosets.

Hint: To define a map on the set of cosets the usual approach is to define a map $\varphi : G \to G$ and show that if $aH = bH$ the $H\phi(a) = H\phi(b)$.

Remark: This is obvious if G is finite but the result is true for infinite groups G.

(5) Let $(a_1, \ldots, a_n) \in \prod_{i=1}^n G_i$. Suppose that a_i has finite order r_i in G_i for each i. Prove that $(a_1, \ldots, a_n)$ has finite order in $\prod_{i=1}^n G_i$ equal to the least common multiple of the r_i, $\text{lcm}(r_1, \ldots, r_n)$.

(6) Let $r_1, \ldots, r_m \in \mathbb{Z}^+$. Define a subgroup L of $\mathbb{Z}$ by

$$L = \cap_{i=1}^m r_i\mathbb{Z}.$$

Show that L is a nontrivial subgroup of $\mathbb{Z}$. Thus L has a unique positive generator $l \in \mathbb{Z}^+$. Show that $l = \text{lcm}(r_1, \ldots, r_m)$.