First Mid-Term for M365C

- This is an exam. You may consult your notes and textbooks but MUST NOT consult other people.
- Please write neatly on only 1 side of the sheet of paper.
- Please use a separate sheet for each question.
- All problems are of equal value though not necessarily equal difficulty.

Due Wednesday March 9th

(1) Let $C_0 = [0,1]$. We get C_1 by deleting the open interval $(1/3, 2/3)$ to obtain

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

To get C_2 we delete the open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ from C_1 to obtain

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

We obtain C_n by deleting the open middle third from each of the closed intervals in C_{n-1}.

We define

$$C = \bigcap_{n=0}^{\infty} C_n.$$

This is called the middle third Cantor set. It is a closed subset of the unit interval.

(a) Find an expression for the sum of the lengths of the intervals in C_n. Hence determine the length of C.

(b) Show that $x \in C$ if and only if x has a ternary expansion

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

with $x_n \neq 1$ for all n.

(c) Prove that C and $[0,1]$ have the same cardinality. Hint: Consider binary expansions for points in $[0,1]$ - paying close attention to the duplicate expansions - use Cantor-Bernstein

(2) Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

(a) Prove that if $\lim_{n\to\infty} a_n = A$ then
$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = A.$$

(b) Find a divergent sequence $(a_n)_{n=1}^{\infty}$ such that
$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} \text{ exists.}$$

(3) Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

(a) Prove that if
$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| < +\infty$$
then $(a_n)_{n=1}^{\infty}$ converges.

(b) Find a convergent sequence $(a_n)_{n=1}^{\infty}$ such that
$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| = +\infty.$$

(4) Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in $\mathbb{R}$. Let $s_n = \sum_{k=1}^{n} b_k$.

(a) Prove the *summation by parts* formula
$$\sum_{k=n}^{N} a_k b_k = \sum_{k=n}^{N-1} (a_k - a_{k+1}) s_k + a_N s_N - a_n s_{n-1}$$
Hint: $b_k = s_k - s_{k-1}$.

(b) Prove that if
$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| < +\infty$$
and
$$\sum_{n=1}^{\infty} b_n \text{ converges}$$
then
$$\sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

(c) Find examples of sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge but such that $\sum_{n=1}^{\infty} a_n b_n$ diverges.