SECOND "MID"-TERM FOR M365C

- This is an exam. You may consult your notes and textbooks but MUST NOT consult other people.
- Please write neatly on only 1 side of the sheet of paper.
- Please use a separate sheet for each question and staple all of the sheets together.
- All problems are of equal value though not necessarily equal difficulty.

DUE FRIDAY MAY 6TH AT 10 AM

(1) Given a function $g : [a, b] \to \mathbb{R}$ we say f is Riemann-Stieltjes integrable with respect to g if there exists an $I \in \mathbb{R}$ with the property that for every $\epsilon > 0$ there exists a partition π of $[a, b]$ such that for all partitions $\pi' \geq \pi$ and all samples σ' with respect to π' we have

$$\left| I - S_g(f, \pi', \sigma') \right| < \epsilon.$$

If g is non-decreasing then by Homework 11 the crucial monotonicity property still holds and we can still prove:

Theorem. *If for all* $\epsilon > 0$ *there exists* π *a partition of* $[a, b]$ *such that* $\overline{S}_g(f, \pi) - \underline{S}_g(f, \pi) < \epsilon$ *then* f *is Riemann-Stieltjes integrable with respect to* g*.*

(a) Let D be an open interval containing $[a, b]$. Suppose that $g : D \to \mathbb{R}$ is non-decreasing and differentiable with g' continuous . Show that if f is Riemann integrable on $[a, b]$ then f is Riemann-Stieltjes integrable with respect to g (Homework 11) *i.e.* for all and

$$\int_a^b f\, dg = \int_a^b f\, g'.$$

Hint: The right hand side exists since both f and g' are Riemann integrable. Using the mean value theorem we get $g(x_i) - g(x_{i-1}) = g'(\xi_i)$ fro some $\xi \in (x_{i-1}, x_i)$. Using uniform continuity of g' you can prove that if $\mu(\pi) < \delta$ then

$$\left| S(fg', \pi, \sigma) - S_g(f, \pi, \sigma) \right| < K\epsilon(b - a)$$

where K is a bound for f.

(b) Consider the Heaviside function

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}.$$

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous from the left at 0 *i.e.* $\lim_{y \to 0^-} f(y) = f(0)$. Prove that

$$\int_{-1}^{1} f dg = f(0).$$

(2) Consider the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) \qquad x(t_0) = x_0$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(a) Using the fundamental theorem of calculus show that if $x(t)$ is a solution to the initial value problem then

$$x(t) = \int_{t_0}^{t} f\big(x(s), s\big) ds + x_0.$$

(b) Suppose that there exists L such that

$$|f(x_1,t) - f(x_2,t)| \leq L|x_1 - x_2|$$

for all $x_1, x_2, t \in \mathbb{R}$. Define an integral operator

$$I(\varphi)(t) = \int_{t_0}^{t} f(\varphi(s), s) ds + x_0$$

Prove that if $\delta > 0$ is chosen sufficiently small then I is a contraction map on $C\big([t_0 - \delta, t_0 + \delta], \mathbb{R}\big)$ *i.e.* show that there exits $\lambda < 1$ such that

$$d_0\big(I(\varphi_1), -I(\varphi_2)\big) \leq \lambda d_0\big(\varphi_1, \varphi_2\big)$$

where $d_0(f,g) = \sup\{|f(t) - g(t)| : t \in [t_0 - \delta, t_0 + \delta]\}$.

(c) Hence, using the contraction mapping principle (Homework 7), show that there exists a unique solution to the initial value problem

$$\frac{dx}{dt} = f(x,t), \qquad x(t_0) = x_0$$

defined for $t \in [t_0 - \delta, t_0 + \delta]$.

Remark: This is called Picard's Theorem. In reality it is only necessary that the function f be continuous and satisfy such a Lipshitz condition locally though then the proof is slightly more tricky. If we only assume continuity then you can still show that a solution exists though it need not be unique. This is called Peano's Theorem and uses the Ascoli-Arzela theorem.

(3) Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}(x)$, the n-th derivative of f at x, is continuous on $[a,b]$ and differentiable on (a,b). Let $x_0 \in (a,b)$ and define the n-th order Taylor polynomial about x_0 by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Show that for each $x \in [a,b]$ there exists ξ between x and x_0 such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

This is known as the Lagrange form of the remainder.
Hint: Consider $g(t) = f(t) - P_n(t) - C(t-x_0)^{n+1}$ where C is chosen so that $g(x) = 0$. Notice that

$$g(x_0) = g'(x_0) = \cdots = g^{(n)}(x_0) = 0$$

from the definition of the Taylor polynomial. Since $g(x) = g(x_0) = 0$ we can iteratively apply Rolles' Lemma.

(4) Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n+1)}(x)$, the $n+1$-th derivative of f at x, is defined and continuous for all $x \in [a,b]$. Let $x_0 \in (a,b)$ and define the n-th order Taylor polynomial about x_0 by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Show that

$$f(x) = P_n(x) + \int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt$$

This is often known as the integral form of the remainder.
Hint: Proceed by integration by parts.
Remark: Notice that the hypotheses of this theorem are stronger than the hypotheses of the previous one. Under these stronger hypotheses you can obtain the previous theorem from this one via the mean value theorem for integrals.