SPRING 2005 MATH 365C FINAL

Print your name and UT ID number *neatly* on each sheet of paper.

Part A : Answer *ALL* questions

Each question is worth 5 points.

(1) Define what it means for $x \in \mathbb{R}$ to be the limit of a sequence $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$.
(2) Define the limes superior ($\limsup$) of $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$.
(3) Define the notions of absolute convergence and conditional convergence for infinite series $\sum_{n=1}^{\infty} a_n$. Give an example of a conditionally convergent series.
(4) Give the definition of a metric space.
(5) Define what it means for a set to be open in a metric space (X, d). Give an example of a set which is simultaneously open and closed.
(6) Define the interior of a set $A \subset X$ in a metric space (X, d).
(7) Define what it means for a metric space (X, d) to be complete.
(8) Define what it means for a function $f : (X, d_X) \to (Y, d_Y)$ to be continuous at $x \in X$.
(9) Define what it means for a function $f : (X, d_X) \to (Y, d_Y)$ to be uniformly continuous.
(10) Define what it means for a sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n : [a, b] \to \mathbb{R}$ to converge uniformly to a function $f : [a, b] \to \mathbb{R}$.
(11) Define what it means for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$ with integral I.
(12) Define what it means for $f : (a, b) \to \mathbb{R}$ to be differentiable at $x \in (a, b)$.

Part B : Give proofs of *TWO* theorems from *EACH* of the *THREE* sections for a total of *SIX* proofs

Each proof is worth 20 points.

If more than two proofs are attempted in a section then the *BEST* two will be used to determine the grade for that section.

You may use other results from the course without proof but you must give *COMPLETE* statements of any results that you use.

1. REAL SEQUENCES AND SERIES

(1) If $(a_n)_{n=1}^{\infty}$ is a non-decreasing sequence in $\mathbb{R}$ then either $(a_n)_{n=1}^{\infty}$ is convergent or $\lim_{n\to\infty} a_n = +\infty$.

(2) Let $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $A \in \mathbb{R}$. The following statements are equivalent
  (a) $\limsup_{n\to\infty} a_n = A$
  (b) For every $B > A$ we have $a_n < B$ for all but finitely many n. For every $B < A$ we have $a_n > B$ for infinitely many n.

(3) Let $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ be a real sequence and let $\lambda = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. If $\lambda < 1$ then $\sum_{n=1}^{\infty} a_n$ converges. If $\lambda > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

(4) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

2. METRIC SPACES AND CONTINUITY

(1) A closed and bounded interval $[a,b] \subset \mathbb{R}$ is compact.

(2) If $[a,b]$ is closed and bounded, and $f : [a,b] \to \mathbb{R}$ is continuous then f is uniformly continuous.

(3) If (X,d) is a compact metric space and $f : (X,d) \to (\mathbb{R}, |\cdot|)$ is continuous then there exists $x \in X$ such that $f(y) \leq f(x)$ for all $y \in X$.

(4) If $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(x) \in \mathbb{R}$ then $g \circ f : \mathbb{R} \to \mathbb{R}$ is continuous at x.

3. INTEGRATION AND DIFFERENTIATION

1. If $f : [a, b] \to \mathbb{R}$ is monotone (i.e. non-increasing or non-decreasing) function then f is Riemann integrable on $[a, b]$.
2. If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are Riemann integrable on $[a, b]$ then the product $f \cdot g$ is Riemann integrable.
3. If $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are differentiable at $x \in (a, b)$ the $f(t) \cdot g(t)$ is differentiable at x and
$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$
4. If $x \in (a, b)$ is a local maximum or minimum for $f : (a, b) \to \mathbb{R}$ and f is differentiable at x then $f'(x) = 0$.