

HOMEWORK 10 FOR M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE WEDNESDAY APRIL 20TH

- (1) Let (X, d) be a compact metric space. Suppose $(f_n)_{n=1}^\infty \subseteq C(X, \mathbb{R})$.
- (a) Suppose that $M_n \geq 0$ is such that $|f_n(x)| \leq M_n$ for all $x \in X$. Show that if

$$\sum_{n=1}^{\infty} M_n < +\infty$$

then the infinite sum

$$f(x) = \sum_{n=1}^{\infty} f_n(x) := \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n(x)$$

is a continuous function on X .

- (b) Hence show that e^x is a continuous function on \mathbb{R} .
- (2) Each of the following functions is defined on $(0, 1]$. Extend each to a function $f : [0, 1] \rightarrow \mathbb{R}$ by setting $f(0) = 0$. For each function decide whether it is Riemann integrable on $[0, 1]$ and give a proof.
- (a) $\sin^2\left(\frac{1}{x}\right)$
 (b) $\frac{1}{x} \sin\left(\frac{1}{x}\right)$.
 (c) $\ln(x)$.
 (d) $\frac{\sin(x)}{x}$.

- (3) Define a function $F : C([a, b], \mathbb{R}) \rightarrow C([a, b], \mathbb{R})$ by

$$(F\phi)(x) = \int_a^x \phi(y) dy.$$

- (a) Show that F is continuous.
 (b) Show that for any $\phi \in C([a, b], \mathbb{R})$ there exists $L > 0$ such that

$$|(F\phi)(x) - (F\phi)(y)| \leq L|x - y|.$$

- (c) Show that if $\Phi \subseteq C([a, b], \mathbb{R})$ is uniformly bounded then $F(\Phi)$ is precompact. [Hint: Apply Ascoli-Arzelà].

- (4) Suppose that $f_n : [a, b] \rightarrow \mathbb{R}$ is a sequence of functions which converges uniformly to a function $f : [a, b] \rightarrow \mathbb{R}$ and suppose that each f_n is Riemann integrable over $[a, b]$. Show that f is Riemann integrable and that

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n.$$

Remark: We do not assume that f_n is continuous. If each f_n was continuous then the uniform limit f would be continuous and hence Riemann integrable.