Homework 11 for M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Wednesday April 27th

(1) Given functions $f, g : [a, b] \to \mathbb{R}$ we define the Riemann-Stieltjes sum of f with respect to partition $\pi = \{x_i\}_{i=0}^n$ and sample $\sigma = \{\xi_i\}_{i=1}^n$ to be

$$S_g(f, \pi, \sigma) = \sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1}))$$

and then

$$\overline{S}_g(f, \pi) = \sup\{S_g(f, \pi, \sigma) : \sigma \text{ associated to } \pi\}$$

(a) Show that if g is non-decreasing then

$$\overline{S}_g(f, \pi') \leq \overline{S}_g(f, \pi)$$

for $\pi \leq \pi'$.

(b) Show that if g is not non-decreasing then it is not necessarily true that

$$\overline{S}_g(f, \pi') \leq \overline{S}_g(f, \pi)$$

for $\pi \leq \pi'$. Hint : Consider the effect of intervals $[x_{i-1}, x_i]$ with $g(x_i) - g(x_{i-1}) < 0$.

*Remark:*If g is a a non-decreasing function then we can build the theory of Riemann-Stieltjes integration in the same way as we built the theory of Riemann integration. This is good for probability theory since g can be taken to be the cumulative distribution function for a probability distribution. In the case we have a probaility density function g' then we have

$$\text{(Riemann-Stieltjes)} \int_a^b f\, dg = \int_a^b f\, g' \text{ (Riemann)}$$

(2) Let $f, g : I \to \mathbb{R}$ for some interval $I \subset \mathbb{R}$. Prove that if f and g are differentiable at x then $f \cdot g$ is differentiable at x and

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

(3) Show that for $n \in \mathbb{N}$

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}).$$

Let $n \in \mathbb{Z}$ and define $f(x) = x^n$. Prove that f is differentiable on $\mathbb{R}$ (or $\mathbb{R}\backslash\{0\}$ if $n < 0$) and show that $f'(x) = nx^{n-1}$.

(4) Show that if $E : \mathbb{R} \to \mathbb{R}$ satisfies $E(x+y) = E(x) \cdot E(y)$ then E is differentiable at x if and only if it is differentiable at 0 and then $E'(x) = E(x) \cdot E'(0)$.