Homework 5 for M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Wednesday March 2nd at 10 am

(1) [Root Test] Use the comparison test with a geometric series to prove that if $(a_n)_{n=1}^{\infty}$ is a sequence in R and

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges. Similarly show that if

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

(2) [Cauchy Condensation Test] If $(a_n)_{n=1}^{\infty}$ is a non-increasing sequence, with $a_n > 0$ for each n then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Hint: Show that if we let $s_n = \sum_{i=1}^{n} a_i$ then

$$\frac{1}{2}\sum_{i=1}^{n} 2^i a_{2^i} \leq s_{2^n} - s_1 \leq \sum_{i=1}^{n} 2^{i-1} a_{2^{i-1}}$$

This estimate is similar to the one required for the harmonic series in the previous homework

(3) Use this test to show that the series $\sum_{n=1}^{\infty} n^{-s}$ converges if and only if $s > 1$.

(4) Show that if $(a_n)_{n=1}^{\infty}$ is non-increasing with $\lim_{n\to\infty} a_n = 0$ then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

This shows that the *alernating harmonic series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is convergent. Since the *harmonic series* is not convergent this gives an example of a conditionally convergent series.