Homework 6 for M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Wednesday March 23rd at 10 am

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$.
   (a) Show that if f is continuous at (at least) one point then f is continuous on all of $\mathbb{R}$.
   (b) Let $c = f(1)$. Show that $f(q) = cq$ for all $q \in \mathbb{Q}$.
   (c) Hence prove that if f is continuous at (at least) one point then $f(x) = cx$.

(2) (a) Prove that any polynomial of odd degree has at least one real root.
   (b) Prove that if I is an open interval and $f : I \to \mathbb{R}$ satisfies, for all $a, b \in I$ and all $0 \leq t \leq 1$
   $$f\big((1-t)a + tb\big) \leq (1-t)f(a) + tf(b),$$
   then f is continuous.
   Such a function f is called *convex*. For example the function $f(x) = x^2$ is convex.
   (c) Find a discontinuous convex function $f : [0, 1] \to \mathbb{R}$.

(3) Let I be a bounded but not necessarily closed interval
   (a) Prove that if $f : I \to \mathbb{R}$ is *uniformly* continuous on I then f is bounded.
   (b) Prove that if $I = (a, b)$ and $f : I \to \mathbb{R}$ is *uniformly* continuous on I then $\lim_{t\to a^+} f(t)$ and $\lim_{t\to b^-} f(t)$ exist. Hence show that you can extend f to a continuous function on $[a, b]$.
   (c) Show that if $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence in I and f is uniformly continuous on I then $\big(f(x_n)\big)_{n=1}^{\infty}$ is a Cauchy sequence in $f(I)$.

(4) Show that $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$ with the distance
$$d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$
is a metric space.