Homework 7 for M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Wednesday March 30th at 10 am

(1) Let (X, d) be a metric space. Prove that:
   (a) $\varnothing$ and X are open.
   (b) If $\{U_\alpha : \alpha \in I\}$ is a set of open sets then $\bigcup_{\alpha \in I} U_\alpha$ is an open set.
   (c) If $\{U_i\}_{i=1}^n$ is a finite set of open sets then $\bigcap_{i=1}^n U_i$ is an open set.
   (d) Give an example to show that a countable intersection of open sets need not be open.

(2) Let (X, d) be a metric space.
   (a) Let $C \subseteq X$ be a closed subset. Prove that if (x_n) is a convergent sequence in (X, d) with $x_n \in C$ for all n then $\lim_{n\to\infty} x_n \in C$.
   (b) Let (X, d) be a metric space. Let $A \subseteq X$ be a subset. Prove that if for all convergent sequences (x_n) with $x_n \in C$ for all n we have $\lim_{n\to\infty} x_n \in A$ then A is a closed subset. [Hint: Prove the contrapositive: if A is not closed then there exists ...]

(3) Let (X, d) be a complete metric space and suppose $f : X \to X$ is a function with the property that there exists $\lambda < 1$ such that

$$d\big(f(x), f(y)\big) < \lambda d(x, y).$$

   Prove that there exists a unique point $x \in X$ such that $f(x) = x$. [Hint: Pick $x_0 \in X$ and let $x_{n+1} = f(x_n)$. Show the sequence is Cauchy and hence converges and that the limit must be a fixed point. This apparently simple theorem is in fact immensely powerful.]

(4) Let (X, d) be a compact metric space.
   (a) Prove that if $C \subseteq X$ is closed then (C, d) is a compact metric space.
   (b) Prove that if $f : X \to \mathbb{R}$ is continuous then there exist $x_{\max} \in X$ and $x_{\min} \in X$ such that $f(x_{\max}) = \sup\{f(x) : x \in X\}$ and $f(x_{\min}) = \inf\{f(x) : x \in X\}$.