HOMEWORK 8 FOR M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE WEDNESDAY APRIL 6TH AT 10 AM

(1) Let (X, d) be a metric space and let $A \subseteq X$. Prove that

$$(A^\circ)^c = \overline{(A^c)}$$

and

$$(\overline{A})^c = (A^c)^\circ$$

(2) Show that if (X, d) is a sequentially compact metric space then for any sequence of nested closed sets (A_n)

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

(3) Prove that if $f : (X, d_X) \to (Y, d_Y)$ is continuous and (X, d_x) is compact then $f(X)$ is a compact subset of Y i.e. $\big(f(X), d_Y\big)$ is a compact metric space.

(4) Let X be a set endowed with two metrics d_1 and d_2. Suppose that there exists $C_1, C_2 > 0$ such that

$$C_1 d_1(x, y) \leq d_2(x, y) \leq C_2 d_1(x, y)$$

for all $x, y \in X$. Such metrics are called uniformly equivalent.

(a) Prove that a seqence (x_n) converges to x in the d_1 metric if and only if (x_n) converges to x in the d_2 metric.

(b) Consider the two metrics

$$d_1\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$d_2\big((x_1, y_1), (x_2, y_2)\big) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

on $\mathbb{R}^2$. Describe the geometry of the open balls in each metric and then prove they are uniformly equivalent.

Indeed both metrics arise from norms on $\mathbb{R}^2$ and all norms on finite dimensional vector spaces are uniformly equivalent.