

HOMWORK 9 FOR M365C

- Please label your homework clearly with your name.
- Homework must be neatly written on one side of the paper only and should be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE WEDNESDAY APRIL 13TH AT 10 AM

- (1) We say $A \subseteq (X, d)$ is a totally bounded subset of a metric space (X, d) if for every $\epsilon > 0$ there exists a finite collection of points $\{x_1, \dots, x_n\} \subseteq X$ such that

$$A \subseteq \bigcup_{i=1}^n B(x_i, \epsilon).$$

- (a) Prove that if $A \subseteq (X, d)$ is totally bounded then \overline{A} is totally bounded. [Hint: If $\{x_1, \dots, x_n\} \subseteq X$ is an ϵ -net for A then it is not necessarily an ϵ -net for \overline{A} . It will be an ϵ' -net for \overline{A} for any $\epsilon' > \epsilon$.]
- (b) Prove that if A is a totally bounded subset of (X, d) then for each $\epsilon > 0$ the finite collection of points can be chosen to be from A i.e. for every $\epsilon > 0$ there exists a finite collection of points $\{a_1, \dots, a_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^n B(a_i, \epsilon).$$

Remark: Thus if $A \subseteq (X, d)$ is a totally bounded set then (A, d) is a totally bounded metric space.

- (c) Hence prove that if A is a totally bounded subset of a complete metric space (X, d) then (\overline{A}, d) is a compact metric space.
- (2) Let (X, d_X) and (Y, d_Y) are metric spaces and let $f : X \rightarrow Y$ be continuous. Show that if (X, d) is connected then $f(X)$ is connected.
- (3) (a) Let $\pi = (x_i)_{i=0}^n$ be a partition of $[-a, a]$. Show that there exists a partition $\pi' = (y_j)_{j=0}^{2m}$ with the properties
- Symmetry:** for all $0 \leq j \leq 2m$ $y_j = -y_{2m-j}$.
- Refinement:** $\pi \leq \pi'$.
- (b) Suppose $f : [-a, a] \rightarrow \mathbb{R}$ is odd, i.e. $f(-x) = -f(x)$. Suppose that $\int_{-a}^a f$ exists. Show that $\int_{-a}^a f = 0$ using the definition of the Riemann integral.
- (c) Suppose $f : [-a, a] \rightarrow \mathbb{R}$ is even, i.e. $f(-x) = f(x)$. Suppose that $\int_{-a}^a f$ exists. Show that $\int_{-a}^a f = 2 \int_0^a f$ using the definition of the Riemann integral.

(4) Show that for $f : [a, b] \rightarrow \mathbb{R}$, $g : [a, b] \rightarrow \mathbb{R}$, and $\alpha, \beta \in \mathbb{R}$ we have

$$S(\alpha f + \beta g, \pi, \sigma) = \alpha S(f, \pi, \sigma) + \beta S(g, \pi, \sigma)$$

for all partitions π of $[a, b]$ and all samples associated to π . Hence show that if f and g are Riemann integrable on $[a, b]$ then $\alpha f + \beta g$ is Riemann integrable and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$