1. Metric Spaces

Definition 1. *A metric space* (X, d) *is a set* X *with a function* $d : X \times X \to \mathbb{R}$ *with the properties*

1. $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Definition 2 (Limit of a Sequence I). *Let* (X, d) *be a metric space and* $(x_n)_{n=1}^{\infty} \subseteq X$ *a sequence. We say* $x \in X$ *is the limit of* $(x_n)_{n=1}^{\infty}$, *written* $\lim_{n\to\infty} x_n = x$ *if for every* $\epsilon > 0$ *there exists* $N_\epsilon > 0$ *such that for all* $n > N_\epsilon$ $d(x_n, x) < \epsilon$.

If a sequence has a limit we call the sequence convergent.

Definition 3 (Limit of a Sequence II). *Let* (X, d) *be a metric space and* $(x_n)_{n=1}^{\infty} \subseteq X$ *a sequence. We say* $x \in X$ *is the limit of* $(x_n)_{n=1}^{\infty}$, *written* $\lim_{n\to\infty} x_n = x$ *if* $lim_{n\to\infty} d(x_n, x) = 0$.

Definition 4 (Open Balls). *Let* (X, d) *be a metric space,* $x \in X$, *and* $r > 0$. *The open ball of radius* r *around* x, *denoted* $B(x, r)$, *is*

$$B(x, r) := \{y \in X : d(x, y) < r\}$$

Definition 5 (Closed Balls). *Let* (X, d) *be a metric space,* $x \in X$, *and* $r > 0$. *The closed ball of radius* r *around* x, *denoted* $\overline{B}(x, r)$, *is*

$$\overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$$

Definition 6. *Let* (X, d) *be a metric space and* $U \subseteq X$. *We say* U *is open of for all* $x \in U$ *there exists* $r_x > 0$ *such that* $B(x, r_x) \subseteq U$.

Definition 7. *Let* (X, d) *be a metric space and* $F \subseteq X$. *We say* F *is closed if* F^c *is open.*

Definition 8 (Limit of a Sequence III). *Let* (X, d) *be a metric space and* $(x_n)_{n=1}^{\infty} \subseteq X$ *a sequence. We say* $x \in X$ *is the limit of* $(x_n)_{n=1}^{\infty}$, *written* $\lim_{n\to\infty} x_n = x$ *if for every* $\epsilon > 0$ *there exists* $N_\epsilon > 0$ *such that for all* $n > N_\epsilon$ $x_n \in B(x, \epsilon)$.

Theorem 1 (Properties of Open Sets). *Let* (X, d) *be a metric space.*

1. *The sets* $\varnothing$ *and* X *are open.*
2. *If* $\{U_i\}_{i=1}^{n}$ *is a finite collection of open sets then* $\cap_{i=1}^{n} U_i$ *is open.*
3. *If* $\{U_\alpha\}_{\alpha \in I}$ *is any collection of open sets then* $\cup_{\alpha \in I} U_i$ *is open.*

Theorem 2 (Properties of Closed Sets). *Let* (X, d) *be a metric space.*

1. *The sets* $\varnothing$ *and* X *are closed.*

(2) *If* $\{F_i\}_{i=1}^n$ *is a finite collection of closed sets then* $\cup_{i=1}^n F_i$ *is closed.*
(3) *If* $\{F_\alpha\}_{\alpha\in I}$ *is any collection of closed sets then* $\cap_{\alpha\in I} F_i$ *is closed.*

Definition 9 (Closure)**.** *Let* (X,d) *be a metric space and* $A \subseteq X$*. The closure of* A*, denoted* $\overline{A}$*, is*

$$\overline{A} = \cap\{F : A \subseteq F, F \text{ closed }\}.$$

The closure of A is the smallest closed set containing A.

Theorem 3. *Let* (X,d) *be a metric space. For all* $x \in X$ *and* $r > 0$

$$\overline{B(x,r)} \subseteq \overline{B}(x,r).$$

Definition 10 (Interior)**.** *Let* (X,d) *be a metric space and* $A \subseteq X$*. The interior of* A*, denoted* A°*, is*

$$A^\circ = \cup\{U : U \subseteq A, U \text{ open }\}.$$

The interior of A is the largest open set contained in A.

Theorem 4. *Let* (X,d) *be a metric space and* $A \subseteq X$*.*
(1) $x \in \overline{A}$ *if and only if for every* $r > 0$ $B(x,r) \cap A \neq \varnothing$*.*
(2) $x \in A^\circ$ *if and only if there exists* $r_x > 0$ *such that* $B(x,r) \subseteq A$*.*

Definition 11. *Let* (X,d) *be a metric space and* $A \subseteq X$*. A point* $x \in X$ *is called a limit point of* A *if for every* $r > 0$

$$B(x,r) \cap A \not\subseteq \{x\}.$$

A point $x \in A$ *that is not a limit point is called an isolated point.*

Theorem 5 (Sequential Definition of Closure)**.** *Let* (X,d) *be a metric space and* $A \subseteq X$*.* $x \in \overline{A}$ *if and only if there exists a sequence* $(x_n)_{n=1}^\infty \subseteq A$ *with* $\lim_{n\to\infty} x_n = x$*.*

Definition 12 (Dense)**.** *Let* (X,d) *be a metric space. A subset* $A \subseteq X$ *is called dense if* $\overline{A} = X$*.*

If A is dense then every point in X is the limit of a sequence in A.

Definition 13 (Nowhere Dense)**.** *Let* (X,d) *be a metric space. A subset* $A \subseteq X$ *is called nowhere dense if* $\overline{A}^\circ = \varnothing$*.*

1.1. **Cauchy Sequences and Completeness.**

Definition 14 (Cauchy Sequence)**.** *Let* (X,d) *be a metric space. A sequence* $(x_n)_{n=1}^\infty$ *is called a Cauchy sequence if for every* $\epsilon > 0$ *there exists* $N > 0$ *such that for all* $m,n > N$ $d(x_n,x_m) < \epsilon$*.*

Theorem 6 (Convergent Sequences are Cauchy)**.** *Let (X, d) be a metric space. If $(x_n)_{n=1}^{\infty}$ is convergent then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.*

Definition 15 (Complete)**.** *A metric space (X, d) is called a complete metric space if every Cauchy sequence is convergent.*

Our basic example of a complete metric space is $(\mathbb{R}, |\cdot|)$.

Theorem 7. *For $1 \leq p \leq \infty$ ℓ_p is a complete metric space.*

1.2. **Compactness.**

Definition 16 (Compactness)**.** *A metric space (X, d) is called (sequentially) compact if every sequence has a convergent subsequence.*

Definition 17. *Let (X, d) be a metric space. A subset $A \subseteq X$ is called compact if the induced metric space (A, d) is compact.*

Theorem 8. *Let (X, d) be a metric space. If $A \subseteq X$ is compact then A is closed.*

Theorem 9. *Let (X, d) be a compact metric space. If $A \subseteq X$ is closed then A is compact.*

Theorem 10 (Bolzano-Weierstraß)**.** *The metric space $\big([a, b], |\cdot|\big)$ is compact.*

Theorem 11 (Nested Ball Theorem)**.** *A metric space (X, d) is complete if and only if every sequence of closed balls $\big(\overline{B}(x_n, r_n)\big)_{n=1}^{\infty}$ satisfying*

(1) $\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n)$.
(2) $\lim_{n\to\infty} r_n = 0$.

has

$$\bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n) \neq \varnothing.$$

Theorem 12 (Baire Category Theorem)**.** *A complete metric space (X, d) cannot be written as a countable union of nowhere dense sets.*

1.3. **Normed Linear Spaces.**

Definition 18. *A set V is called a vector space if it is endowed with operations*

$$+ : V \times V \to V$$
$$\cdot : \mathbb{R} \times V \to V$$

that satisfy

(1) *for all* $\vec{u}, \vec{v} \in V$

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}.$$

(2) *for all* $\vec{u}, \vec{v}, \vec{w} \in V$

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}.$$

(3) *there exists a unique* $\vec{0} \in V$ *such that for all* $\vec{v} \in V$

$$\vec{0} + \vec{v} = \vec{v}.$$

(4) *for every* $\vec{v} \in V$ *there exists a unique* $\vec{-v} \in V$ *such that*

$$\vec{v} + \vec{-v} = \vec{0}.$$

(5) *for all* $\vec{v} \in V$ *and all* $\alpha, \beta \in \mathbb{R}$

$$\alpha \cdot (\beta \cdot \vec{v}) = (\alpha\beta) \cdot \vec{v}.$$

(6) *for all* $\vec{v} \in V$

$$1 \cdot \vec{v} = \vec{v}.$$

(7) *for all* $\vec{v} \in V$ *and all* $\alpha, \beta \in \mathbb{R}$

$$(\alpha + \beta) \cdot \vec{v} = \alpha \cdot \vec{v} + \beta \cdot \vec{v}.$$

(8) *for all* $\vec{u}, \vec{v} \in V$ *and all* $\alpha \in \mathbb{R}$

$$\alpha \cdot (\vec{u} + \vec{v}) = \alpha \cdot \vec{u} + \alpha \cdot \vec{v}.$$

In fact uniqueness of the elements $\vec{0}$ and $\vec{-v}$ can be proved and does not have to be part of the definition.

Definition 19 (Norm)**.** *Let* V *be a vector space. A non-negative function* $n : V \to \mathbb{R}$ *is called a norm if it satisfies*

(1) $n(\vec{v}) = 0$ *if and only if* $\vec{v} = \vec{0}$.
(2) $n(\alpha \cdot \vec{v}) = |\alpha| n(\vec{v})$.
(3) $n(\vec{u} + \vec{v}) \leq n(\vec{u}) + n(\vec{v})$.

As with the definition of metric non-negativity is actually a consequence of properties 1 and 3 and need not be part of the definition. Norms are traditionally denoted $\|\cdot\|$.

Theorem 13. *Let* $(V, \|\cdot\|)$ *be a normed vector space. The function* $d : V \times V \to \mathbb{R}$ *given by*

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

is a metric on V.

Theorem 14. *For every* $1 \leq p \leq \infty$ *the space* ℓ_p *endowed with component-wise addition and scalar multiplication and*

$$\|(x_n)\|_p := \big(\sum_{n=1}^{\infty} |x_n|^p\big)^{\frac{1}{p}}$$

is a normed vector space.

In the process of proving this we meet two important inequalities.

Definition 20. *We call two number* $p, q \in \mathbb{R}$ *dual if*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We also declare 1 *and* ∞ *to be dual.*

Theorem 15 (Hölder's Inequality)**.** *Let* p *and* q *be dual. If* $(x_n) \in \ell_p$ *and* $(y_n) \in \ell_q$ *then*

$$\|(x_n y_n)\|_1 \leq \|(x_n)\|_p \|(y_n)\|_q.$$

If $p, q \in \mathbb{R}$ *are dual then this is the same as*

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \big(\sum_{n=1}^{\infty} |x_n|^p\big)^{\frac{1}{p}} \big(\sum_{n=1}^{\infty} |y_n|^q\big)^{\frac{1}{q}}.$$

Theorem 16 (Minkowski's Inequality)**.** *For* $1 \leq p \leq \infty$

$$\|(x_n + y_n)\|_p \leq \|(x_n)\|_p + \|(y_n)\|_p$$

Theorem 17. *Let* $(V, \|\cdot\|)$ *be a normed vector space. If* $(\vec{u}_n)_{n=1}^{\infty} \subset V$ *and* $(\vec{u}_n)_{n=1}^{\infty} \subset V$ *are convergent sequences then* $(\vec{u}_n + \vec{v}_n)_{n=1}^{\infty}$ *is a convergent sequence and*

$$\lim_{n\to\infty} (\vec{u}_n + \vec{v}_n) = \lim_{n\to\infty} \vec{u}_n + \lim_{n\to\infty} \vec{v}_n.$$

Theorem 18. *Let* $(V, \|\cdot\|)$ *be a normed vector space.If* $(\beta_n)_{n=1}^{\infty}$ *is a convergent sequence in* $(\mathbb{R}, |\cdot|)$ *and* $(\vec{u}_n)_{n=1}^{\infty}$ *is a convergent sequence in* $(V, \|\cdot\|)$ *then* $(\beta_n \cdot \vec{u}_n)_{n=1}^{\infty}$ *is a convergent sequence in* $(V, \|\cdot\|)$ *and*

$$\lim_{n\to\infty} (\beta_n \cdot \vec{u}_n) = (\lim_{n\to\infty} \beta_n) \cdot (\lim_{n\to\infty} \vec{u}_n).$$

2. Riemann Integration

Definition 21 (Partition)**.** *Let* $[a, b]$ *be a closed and bounded interval in* $(\mathbb{R}, |\cdot|)$*. A finite sequence* $(x_i)_{i=0}^{n}$ *is called a partition of* $[a, b]$ *if*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We will normally denote such partitions by π*.*

Definition 22 (Refinement)**.** *Let $[a,b]$ be a closed and bounded interval in $(\mathbb{R}, |\cdot|)$ and $\pi = (x_i)_{i=0}^n$ and $\pi' = (y_i)_{i=0}^m$ be two partitions of $[a,b]$. We say π' is a refinement of π, written $\pi \leq \pi'$, if*

$$\{x_0, \ldots, x_n\} \subseteq \{y_0, \ldots, y_m\}.$$

Definition 23 (Sample)**.** *Let $[a,b]$ be a closed and bounded interval in $(\mathbb{R}, |\cdot|)$ and $\pi = (x_i)_{i=0}^n$ a partition of $[a,b]$. A sample associated to π is a finite sequence $\sigma = \{\xi_i\}_{i=1}^n$ such that $x_{i-1} \leq \xi_i \leq x_i$ for $i = 1, \ldots, n$.*

Definition 24 (Riemann Sum)**.** *Let $f : [a,b] \to \mathbb{R}$, $\pi = (x_i)_{i=0}^n$ be a partition of $[a,b]$, and $\sigma = (\xi_i)_{i=1}^n$ be a sample associated to π. The Riemann sum for f over π with sample σ, denoted $S(f, \pi, \sigma)$, is given by*

$$S(f, \pi, \sigma) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

Definition 25 (Riemann Integral)**.** *Let $f : [a,b] \to \mathbb{R}$. We call f Riemann integrable if there exists $I \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a partition π such that for every partition $\pi' \geq \pi$ and every sample σ associated to π' we have*

$$|S(f, \pi', \sigma) - I| < \epsilon.$$

I is called the integral of f over $[a,b]$ and denoted $\int_a^b f$ or $\int_a^b f(x)dx$.

That the integral is unique follows from the fact that the space of partitions is actually a lattice. For every pair of partitions π and π' we have

$$\pi \vee \pi' = \text{lub}\{\pi, \pi'\} = \pi \cup \pi'$$
$$\pi \wedge \pi' = \text{glb}\{\pi, \pi'\} = \pi \cap \pi$$

Theorem 19. *If $f : [a,b] \to \mathbb{R}$ is Riemann integrable then f is bounded.*

Definition 26 (Upper and Lower Riemann Sums)**.** *Let $f : [a,b] \to \mathbb{R}$ and $\pi = (x_i)_{i=0}^n$ be a partition of $[a,b]$. The upper Riemann sum for f over π, denoted $\overline{S}(f, \pi)$, is given by*

$$\overline{S}(f, \pi) = \sup_{\sigma} S(f, \pi, \sigma)$$

and the lower Riemann sum for f over π, denoted $\underline{S}(f, \pi)$, is given by

$$\underline{S}(f, \pi) = \inf_{\sigma} S(f, \pi, \sigma)$$

Clearly

$$-\infty \leq \underline{S}(f,\pi) \leq \overline{S}(f,\pi) \leq \infty.$$

If $m \leq f(x) \leq M$ for all $x \in [a,b]$ then

$$m(b-a) \leq \underline{S}(f,\pi) \leq \overline{S}(f,\pi) \leq M(b-a).$$

Theorem 20 (Monotonicity)**.** *Let π and π' be partitions of $[a,b]$ with $\pi \leq \pi'$ then for any $f : [a,b] \to \mathbb{R}$*

$$\underline{S}(f,\pi) \leq \underline{S}(f,\pi') \leq \overline{S}(f,\pi') \leq \overline{S}(f,\pi).$$

Theorem 21. *Let $f : [a,b] \to \mathbb{R}$. The function f is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition π of $[a,b]$ such that*

$$\overline{S}(f,\pi) - \underline{S}(f,\pi) < \epsilon.$$

Theorem 22. *Let $f : [a,b] \to \mathbb{R}$. If π_n is a sequence of partition of $[a,b]$ such that*

$$\lim_{n\to\infty} \big(\overline{S}(f,\pi_n) - \underline{S}(f,\pi_n)\big) = 0$$

and σ_n is any sample associated to π_n then

$$\int_a^b f = \lim_{n\to\infty} S(f,\pi_n,\sigma_n).$$

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2.1. **Continuous Functions.**

Definition 27 (Continuity)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x-y| < \delta$ then $\big|f(x), f(y)\big| < \epsilon$. We say f is continuous if it is continuous at all x.*

This is really just a special case of a more general definition for metric spaces.

Definition 28 (Continuity in Metric Spaces)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called continuous at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_X(x,y) < \delta$ then $d_Y\big(f(x), f(y)\big) < \epsilon$. We say f is continuous if it is continuous at all $x \in X$.*

We can rewrite this definition using open balls.

Definition 29 (Continuity in Metric Spaces)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called continuous at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$f\big(B(x,\delta)\big) \subseteq B\big(f(x),\epsilon\big).$$

Finally this can be rewritten in terms of open sets. This particular definition generalizes to topological spaces.

Theorem 23. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $x \in X$ if and only if for every open set $U \subseteq Y$ with $f(x) \in U$ there is an open set $V \subseteq X$ with $x \in V$ such that $f(V) \subset U$. A function $f : X \to Y$ is continuous if and only if for all open sets $U \subseteq Y$ the pre-image $f^{-1}\big(U\big) \subseteq X$ is open.*

Theorem 24 (Sequential Definition of Continuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $x \in X$ if and only if for every convergent sequence $(x_n)_{n=1}^{\infty} \subseteq X$*

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

The definition of continuity allows us to pick different δ at each $x \in X$. If the same $\delta > 0$ works for all $x \in X$ then we have a stronger property.

Definition 30 (Uniform Continuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_X(x, y) < \delta$ then $d_Y\big(f(x), f(y)\big) < \epsilon$.*

Theorem 25 (Extreme Value Theorem)**.** *If (X, d) is a compact metric space and $f : (X, d) \to (\mathbb{R}, |\cdot|)$ is continuous then there exists $x \in X$ such that $f(y) \le f(x)$ for all $y \in X$. Similarly there exists $x \in X$ such that $f(y) \ge f(x)$ for all $y \in X$.*

Theorem 26. *Let (X, d_X) and (Y, d_Y) be metric spaces. If (X, d_X) is compact and $f : X \to Y$ is continuous then f is uniformly continuous.*

Definition 31 (Finite Product Spaces)**.** *Let (X_i, d_i) be a metric space for $i = 1, \dots, n$. The Cartesian product*

$$\prod_{i=1}^{n} X_i = \big\{(x_1, \dots, x_n) : x_i \in X_i\big\}$$

can be made into a metric space by defining

$$d\big((x_i)_{i=1}^n, (y_i)_{i=1}^n\big) = \sum_{i=1}^{n} d_i(x_i, y_i).$$

There are a number of uniformly equivalent metrics for the finite product space. We could have used either

$$d\big((x_i)_{i=1}^n, (y_i)_{i=1}^n\big) = \max_{1\le i\le n} d_i(x_i, y_i)$$
$$d\big((x_i)_{i=1}^n, (y_i)_{i=1}^n\big) = \Big(\sum_{i=1}^n d_i(x_i, y_i)^2\Big)^{\frac{1}{2}}$$

Definition 32 (Countably Infinite Product Spaces)**.** *Let (X_i, d_i) be a metric space for $i \in \mathbb{N}$. The Cartesian product*

$$\prod_{i=1}^\infty X_i = \big\{(x_i)_{i=1}^\infty : x_i \in X_i\big\}$$

can be made into a metric space by defining

$$d\big((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty\big) = \sum_{i=1}^n \frac{\hat{d}_i(x_i, y_i)}{2^n}$$

where $\hat{d}_i(x, y) = \min\{d_i(x, y), 1\}$.

Definition 33 (Projections)**.** *Let I be an index set and let $j \in I$. The projection from $\prod_{i\in I} X_i$ to X_j, denoted π_j, is given by*

$$\pi_j\big((x_i)_{i\in I}\big) = x_j.$$

Theorem 27 (Projections and Sequences)**.** *Let (X_i, d_i) be a metric space for $i \in I$ with I countable. A sequence $(z_n)_{n=1}^\infty \subseteq \prod_{i\in I} X_i$ has*

$$\lim_{n\to\infty} z_n = z$$

in the product space if and only if for all $i \in I$

$$\lim_{n\to\infty} \pi_i(z_n) = \pi_i(z)$$

in (X_i, d_i).

Theorem 28 (Projections and Continuity)**.** *Let (X_i, d_i) be a metric space for $i \in I$ with I countable and let (Y, d_Y) be a metric space. A function $f : (Y, d_Y) \to ()$*

Theorem 29 (Composition of Continuous Functions)**.** *Let (X, d_x), (Y, d_Y), and (Z, d_Z) be metric spaces. If $f : X \to Y$ is continuous at $x \in X$ and $g : Y \to Z$ is continuous at $f(x) \in Y$ then $g \circ f : X \to Z$ is continuous at $x \in X$.*

Theorem 30 (Addition of Continuous Functions)**.** *Let* (X, d_x) *be a metric space and* $(V, \|\cdot\|)$ *a normed vector space. If* $f : X \to V$ *and* $g : X \to V$ *are continuous at* $x \in X$ *then* $f + g : X \to V$ *given by*

$$(f + g)(y) := f(y) + g(y)$$

is continuous at $x \in X$*.*

Theorem 31 (Product of Continuous Functions)**.** *Let* (X, d_x) *be a metric space. If* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *are continuous at* $x \in X$ *then* $f \cdot g : X \to \mathbb{R}$ *given by*

$$(f \cdot g)(y) := f(y) \cdot g(y)$$

is continuous at $x \in X$*.*

Theorem 32 (Division of Continuous Functions)**.** *Let* (X, d_x) *be a metric space. If* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *are continuous at* $x \in X$ *and* $g(x) \neq 0$ *then* $f/g : X \to \mathbb{R}$ *given by*

$$(f/g)(y) := \frac{f(y)}{g(y)}$$

is continuous at $x \in X$*.*

2.2. **Connectedness.**

Definition 34 (Connected)**.** *A metric space* (X, d) *is called disconnected if there are two non-empty open sets* U *and* V *with* $U \cap V = \varnothing$ *and* $U \cup V = X$*. A metric space* (X, d) *is called connected if it is not disconnected.*

Definition 35 (Path Connected)**.** *A metric space* (X, d) *is called path connected if for any two points* $x, y \in X$ *there is a continuous function* $p : ([0, 1], |\cdot\|) \to (X, d)$ *such that* $p(0) = x$ *and* $p(1) = y$*. If* p *can always be chosen to be one-to-one then* (X, d) *is called arc connected.*

Theorem 33. *If the metric space* (X, d) *is path connected then* (X, d) *is connected.*

Theorem 34. *Any interval in* $(\mathbb{R}, |\cdot|)$ *is path connected and these are the only connected subsets of* $(\mathbb{R}, |\cdot|)$*.*

Theorem 35. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. If* (X, d_X) *is connected then* $\big(f(X), d_Y\big)$ *is connected.*

Theorem 36 (Intermediate Value Theorem)**.** *If* $f : ([a, b], |\cdot|) \to (\mathbb{R}, |\cdot|)$ *is continuous and* c *lies between* $f(a)$ *and* $f(b)$ *then there exists* $x \in [a, b]$ *such that* $f(x) = c$*.*

Theorem 37. *The function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there exist continuous functions $g : [a, b \to \mathbb{R}$ and $h : [a, b \to \mathbb{R}$ with $g(x) \le f(x) \le h(x)$ such that*

$$\int_a^b h - \int_a^b g < \epsilon.$$

Definition 36 (Riemann-Stieltjes Sum)**.** *Let $g : [a,b] \to \mathbb{R}$ be a non-decreasing function, $f : [a,b] \to \mathbb{R}$, $\pi = (x_i)_{i=0}^n$ be a partition of $[a,b]$, and $\sigma = (\xi_i)_{i=1}^n$ be a sample associated to π. The Riemann-Steiltjes sum for f over π with sample σ and distribution g, denoted $S_g(f,\pi,\sigma)$, is given by*

$$S_g(f,\pi,\sigma) = \sum_{i=1}^n f(\xi_i)\big(g(x_i) - g(x_{i-1})\big)$$

Definition 37 (Riemann-Stieltjes Integral)**.** *Let $g : [a,b] \to \mathbb{R}$ be a non-decreasing function and $f : [a,b] \to \mathbb{R}$. We call f Riemann-Stelitjes integrable with respect to distribution g if there exists $I \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a partition π such that for every partition $\pi' \ge \pi$ and every sample σ associated to π' we have*

$$|S_q(f,\pi',\sigma) - I| < \epsilon.$$

I is called the Riemann Stieltjes integral of f over $[a,b]$ with respect to the distribution g and denoted $\int_a^b f\,dg$ or $\int_a^b f(x)\,dg(x)$.

Theorem 38. *Let $g : [a,b] \to \mathbb{R}$ be a non-decreasing function. If $f,h : [a,b] \to \mathbb{R}$ are Riemann-Stieltjes integrable on $[a,b]$ with respect distribution g and $\alpha,\beta \in \mathbb{R}$ then $\alpha f + \beta h$ is Riemann-Steiltjes integrable on $[a,b]$ and*

$$\int_a^b (\alpha f + \beta h)\,dg = \alpha \int_a^b f\,dg + \beta \int_a^b h\,dg.$$

Theorem 39. *If $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $m \le f(x) \le M$ then*

$$m(b-a) \le \int_a^b f \le M(b-a)$$

Theorem 40. *Let $g : [a,b] \to \mathbb{R}$ be a non-decreasing function. If $f : [a,b] \to \mathbb{R}$ is Riemann-Steiltjes integrable on $[a,b]$ with respect to distribution g and $m \le f(x) \le M$ then*

$$m\big(g(b) - g(a)\big) \le \int_a^b f\,dg \le M\big(g(b) - g(a)\big)$$

Theorem 41 (Monotonicity of the Riemann Integral)**.** *If* $f, h : [a, b] \to \mathbb{R}$ *are Riemann integrable on* $[a, b]$ *and* $f(x) \leq h(x)$ *then*

$$\int_a^b f \leq \int_a^b h.$$

Theorem 42 (Monotonicity of the Darboux-Stieltjes Integral)**.** *Let* $g : [a, b] \to \mathbb{R}$ *be a non-decreasing function. If* $f, h : [a, b] \to \mathbb{R}$ *are Riemann-Steiltjes integrable on* $[a, b]$ *with respect to distribution* g *and* $f(x) \leq h(x)$ *then*

$$\int_a^b f \, dg \leq \int_a^b h \, dg.$$

Theorem 43. *Let* $g : [a, b] \to \mathbb{R}$ *be a non-decreasing function. If* $f : [a, b] \to \mathbb{R}$ *is Riemann-Steiltjes integrable on* $[a, b]$ *with respect to distribution* g *then*

$$\Big| \int_a^b f \, dg \Big| \leq \int_a^b |f| \, dg.$$

Theorem 44. *If* $f : [a, b] \to \mathbb{R}$ *is Riemann integrable on* $[a, b]$ *then*

$$\Big| \int_a^b f \Big| \leq \int_a^b |f|.$$

Theorem 45 (Mean Value Theorem for Riemann Integrals)**.** *Let* $h : [a, b] \to \mathbb{R}$ *be a non-negative function and* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then there exists* $\xi \in [a, b]$ *such that*

$$\int_a^b f \cdot h = f(\xi) \int_a^b h.$$

Theorem 46 (Mean Value Theorem for Darboux-Stieltjes Integrals)**.** *Let* $g : [a, b] \to \mathbb{R}$ *be a non-decreasing function,* $h : [a, b] \to \mathbb{R}$ *be a non-negative function, and* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then there exists* $\xi \in [a, b]$ *such that*

$$\int_a^b f \cdot h \, dg = f(\xi) \int_a^b h \, dg.$$

Theorem 47. *If* $f, g : [a, b] \to \mathbb{R}$ *are Riemann integrable over* $[a, b]$ *then* $f \cdot g$ *is Riemann integrable over* $[a, b]$*. Furthermore*

$$\Big(\int_a^b f \cdot g \Big)^2 \leq \int_a^b f^2 \cdot \int_a^b g^2.$$

Theorem 48. *Let $a < c < b$ and let $f : [a, b] \to \mathbb{R}$. The function f is Riemann integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. Furthermore in this case*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

If we define

$$\int_a^b f = -\int_b^a f$$

for $b < a$ then we can make sense of

$$\int_a^b f = \int_a^c f + \int_c^b f$$

without assumptions on the relative positions of a, b, and c.

Definition 38 (Mesh)**.** *If $\pi = (x_i)_{i=1}^n$ is a partition of $[a, b]$ then the mesh of π, denoted $\mu(\pi)$, is*

$$\mu(\pi) = \max\{x_i - x_{i-1} : i = 1, \ldots, n\}.$$

Theorem 49 (Monotone Functions are Integrable)**.** *If $f : [a, b] \to \mathbb{R}$ is monotone (i.e. non-increasing or non-decreasing) function then f is Riemann integrable on $[a, b]$. Moreover, if $(\pi_n)_{n=1}^\infty$ is any sequence of partitions of $[a, b]$ with $\mu(\pi_n) \to 0$ as $n \to \infty$ and if $(\sigma_n)_{n=1}^\infty$ is any sequence of samples with σ_n associated to π_n then*

$$\lim_{n\to\infty} S(f, \pi_n, \sigma_n) = \int_a^b f.$$

Theorem 50 (Continuous Functions are Integrable)**.** *If $f : [a, b] \to \mathbb{R}$ is a continuous function then f is Riemann integrable on $[a, b]$. Moreover, if $(\pi_n)_{n=1}^\infty$ is any sequence of partitions of $[a, b]$ with $\mu(\pi_n) \to 0$ as $n \to \infty$ and if $(\sigma_n)_{n=1}^\infty$ is any sequence of samples with σ_n associated to π_n then*

$$\lim_{n\to\infty} S(f, \pi_n, \sigma_n) = \int_a^b f.$$

Theorem 51. *Let $f : [a, b] \to \mathbb{R}$ be bounded and let D denote the set of discontinuities of f. If for every $\epsilon > 0$ there exists a finite collection of intervals, $\{(u_i, v_i)\}_{i=1}^m$, with $D \subseteq \cup_{i=1}^m (u_i, v_i)$ and $\sum_{i=1}^m (v_i - u_i) < \epsilon$ then f is Riemann integrable on $[a, b]$.*

Definition 39. *Let $f : (a, b) \to \mathbb{R}$. We say that f is differentiable at $x \in (a, b)$ if*

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h} \qquad \text{exists.}$$

We write

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Theorem 52. *Let $f : (a,b) \to \mathbb{R}$. The function f is differentiable at $x \in (a,b)$ if and only if there exists a function $\phi : (a,b) \to \mathbb{R}$ that is continuous at x such that*

$$f(y) = f(x) + (y - x)\phi(y)$$

for $y \in (a,b)$. In this case $f'(x) = \phi(x)$.

Theorem 53. *Let $f : (a,b) \to \mathbb{R}$. If f is differentiable at $x \in (a,b)$ then f is continuous at x.*

Theorem 54. *Let $f : (a,b) \to \mathbb{R}$. The function f is differentiable at $x \in (a,b)$ if and only if there exists a linear function $L : (a,b) \to \mathbb{R}$ such that*

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - L(h)|}{|h|} = 0.$$

2.3. **Differential Calculus.**

Theorem 55 (Linearity of the Derivative)**.** *If $f, g : (a,b) \to \mathbb{R}$ are differentiable at $x \in (a,b)$ then $\alpha f + \beta g$ is differentiable at x and*

$$(\alpha f + \beta g)'(x) = \alpha f'(x) + \beta g'(x).$$

Theorem 56 (Product Rule for Differentiation)**.** *If $f, g : (a,b) \to \mathbb{R}$ are differentiable at $x \in (a,b)$ then $f \cdot g$ is differentiable at x and*

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x).$$

Theorem 57 (Chain Rule for Differentiation)**.** *Let $f : (a,b) \to \mathbb{R}$,$g : (c,d) \to \mathbb{R}$, and $f\big((a,b)\big) \subseteq (c,d)$. If f is differentiable at $x \in (a,b)$ and g is differentiable at $f(x) \in (c,d)$ then $g \circ f$ is differentiable at x and*

$$(g \circ f)'(x) = g'\big(f(x)\big)f'(x).$$

Definition 40 (Local Maximum)**.** *Let $f : (a,b) \to \mathbb{R}$. We say f has a local maximum at $c \in (a,b)$ if there is an open set U with $c \in U \subset (a,b)$ such that $f(x) \leq f(c)$ for all $x \in U$.*

Definition 41 (Local Minimum)**.** *Let $f : (a,b) \to \mathbb{R}$. We say f has a local minimum at $c \in (a,b)$ if there is an open set U with $c \in U \subset (a,b)$ such that $f(x) \geq f(c)$ for all $x \in U$.*

Theorem 58. *If $f : (a,b) \to \mathbb{R}$ has a local maximum or local minimum at $c \in (a,b)$ and f is differentiable at c then $f'(c) = 0$.*

Theorem 59 (Rolle's Lemma)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous,f is differentiable for all $x \in (a,b)$, and $f(a) = f(b) = 0$ then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Theorem 60 (Mean Value Theorem)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous and f is differentiable for all $x \in (a,b)$ then there exists $c \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

or equivalently

$$f(b) - f(a) = f'(c)(b - a).$$

Theorem 61. *Let $f : [a,b] \to \mathbb{R}$ be continuous and f be differentiable on (a,b).*

(1) *if $f'(x) = 0$ for all $x \in (a,b)$ the f is constant.*
(2) *if $f'(x) > 0$ for all $x \in (a,b)$ the f is increasing.*
(3) *if $f'(x) \geq 0$ for all $x \in (a,b)$ the f is non-decreasing.*
(4) *if $f'(x) < 0$ for all $x \in (a,b)$ the f is decreasing.*
(5) *if $f'(x) \leq 0$ for all $x \in (a,b)$ the f is non-increasing.*

constant.

2.4. Fundamental Theorems of Calculus.

Definition 42. *Let $f : (a,b) \to \mathbb{R}$. We call $F : (a,b) \to \mathbb{R}$ a primitive or anti-derivative of F if $F'(x) = f(x)$ for all $x \in (a,b)$.*

Theorem 62. *Let $f : (c,d) \to \mathbb{R}$ and F be an antiderivative. If f is integrable on $[a,b] \subset (c,d)$ then*

$$\int_a^b f = F(b) - F(a).$$

Theorem 63. *If $f : [a,b] \to \mathbb{R}$ is integrable and $c \in [a,b]$ then the function $F : [a,b] \to \mathbb{R}$ given by*

$$F(x) = \int_c^x f$$

is continuous. If f is continuous at $x \in (a,b)$ then F is differentiable at x and $F'(x) = f(x)$.

2.5. Integral Calculus.

Theorem 64 (Integration by Parts)**.** *Let $f, g : (c,d) \to \mathbb{R}$ with primitives F, G. If f, g are integrable on $[a,b]$ then*

$$\int_a^b F \cdot g = F(b)G(b) - F(a)G(a) - \int_a^b f \cdot G.$$

Theorem 65 (Substitution Rule)**.** *Let $\phi : (c, d) \to \mathbb{R}$ be differentiable. If ϕ' is integrable on $[\alpha, \beta]$ and f is continuous on $\phi\big([\alpha, \beta]\big)$ then*

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f\big(\phi(y)\big)\phi'(y)\,dy.$$