Math 427K Final Exam

Name:	____________________
UT EID:	____________________

Instructions

- Please put your name and UT EID in the space provided.
- There are 8 questions each worth 10 points and 2 questions worth 15 points. There should be a total of 13 pages including this one and the two tables.
- You have 3 hours to complete the test.
- Please write your working and solutions on the test paper. You may use the back of the pages.
- Calculators are not allowed.

Method of Undetermined Coefficients Table

$g(t)$	$Y(t)$
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0)$
$P_n(t)e^{\alpha t}$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0)e^{\alpha t}$
$P_n(t)e^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$t^s\big[(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0)e^{\alpha t}\cos(\beta t) + (B_n t^n + B_{n-1} t^{n-1} + \cdots + B_0)e^{\alpha t}\sin(\beta t)\big]$

For Instructor's Use

Question 1	______
Question 2	______
Question 3	______
Question 4	______
Question 5	______
Question 6	______
Question 7	______
Question 8	______
Question 9	______
Question 10	______
Total	______

Question 1

Consider a pond that initially contains 20 million liters of unpolluted water. Polluted water containing 10000 kg of PCBs in solution enters the lake at a rate of 4 million liters per year. Water flows out of the lake at the same rate.

1. [4 Points] Write a differential equation for the quantity of PCBs $Q(t)$ in the lake after t years.
2. [5 Points] Find an expression for the quantity of PCBs $Q(t)$ in the lake at time t.
3. [1 Point] Determine the limiting concentration of PCBs in the lake as t goes to $+\infty$.

Question 2 [10 Points]

Solve the ordinary differential equation by finding an appropriate integrating factor

$$(6x^2y + 4xy + 2y^3)dx + (2x^2 + 2y^2)dy = 0.$$

Question 3 [10 Points]

Consider the flow of liquid from a tank with a outlet. Let $h(t)$ denote the height of the surface of the liquid above the outlet. Torricelli's principle states the the rate of change of the height satisfies the differential equation

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}$$

where $A(h)$ is the area of the cross section of the tank at height h, a is the area of the outlet, and α is a constant. For water we take $\alpha = 0.6$. We will assume $g = 10ms^{-2}$.

Consider a water tank of height $4m$ with a cross-section of area $10m^2$ and outlet of area $0.1m$ on the bottom. Suppose that the tank is initially full of water. Find an explicit expression for the height $h(t)$ of the surface of the water above the outlet.

Question 4 [10 Points]

Consider the first order differential equation with the parameter a

$$\frac{dy}{dx} = y(a - y^2)$$

Find and classify all of the equilibrium solutions. Your answer will depend on the parameter a.

Question 5 [10 Points]

For each of the following inhomogeneous constant coefficient second order ordinary differential equation find the *complementary function* and a *suitable form for the particular solution*. You do not need to find the undetermined coefficients.

1. $y'' + 3y' = 3t^4 + t^2 e^{-3t} + \sin(3t)$.
2. $y'' - 4y' + 4y = t^2 - 4te^{2t} + t\sin(t)$.
3. $y'' + 4y = t\sin(2t) + 7\cos(4t)$.

Question 6 [10 Points]

Find the general solution of the inhomogeneous constant coefficient second order differential equation

$$y'' + y = \tan t, \qquad 0 < t < \frac{\pi}{2}.$$

using variation of parameters.

Question 7 [10 Points]

Solve the initial value problem

$$y'' - 2xy' + 12y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

using power series methods.

Question 8 [10 Points]

Solve the inhomogeneous constant coefficient second order linear ordinary differential equation

$$y'' + 2y' + 2y = (1 - u_\pi(t))\cos(t), \quad y(0) = 1, \quad y'(0) = -1$$

using the partial fractions expansion

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{1}{5}\frac{s+2}{s^2+1} - \frac{4}{5}\frac{s+1}{s^2+2s+2}.$$

Question 9 [15 Points]

Suppose the temperature $u(x, t)$ of a metal bar of length 10 cm satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

Suppose that the left hand end of the bar is perfectly insulated and the right hand end is held at $0°C$.

1. [5 points] Write the two ordinary differential equations in $X(x)$ and $T(t)$ which arise from separation of variables and give appropriate boundary conditions. Write the separation constant as $-\lambda$.
2. [7 Points] Solve the eigenvalue problem to find the eigenvalues λ_n and eigenfunctions X_n. You may assume that the eigenfunctions X_n are oscillatory and need not check the other two cases.
3. [3 Points] Give the family of basic solutions to this problem and hence write down the general solution.

Question 10 [15 Points]

Suppose the temperature $u(x,t)$ of a metal bar of length 10 cm satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

Suppose that both ends of the bar are perfectly insulated and that the initial temperature distribution is

$$f(x) = 20 + 10x - x^2.$$

1. [4 Points] Give the basic solutions of the boundary value problem.
2. [9 Points] Find the solution $u(x,t)$ which satisfies the initial condition.
3. [2 Point] What is the asymptotic behavior of the temperature as $t \to \infty$.