Final Exam for M325K

Name:	____________________
UT EID:	____________________

Instructions

- Please put your name and UT EID in the space provided.
- There are 12 questions each worth 10 points.
- You have 3 hours to complete the test.
- Please write your working and solutions on the test paper. You may use the back of the pages.

For Instructor's Use

Question 1	____
Question 2	____
Question 3	____
Question 4	____
Question 5	____
Question 6	____
Question 7	____
Question 8	____
Question 9	____
Question 10	____
Question 11	____
Question 12	____
Total	____

(1) (a) Give the definition of

$$\neg(p \Rightarrow q)$$

using the basic operators $\neg$, $\vee$, and $\wedge$.

(b) Define what it means for a *logical expression* to be a *contradiction* and give a *logical expression* which is a *contradiction*.

(c) Write the argument

$$\begin{array}{rl} & p \vee r \\ & q \vee r \\ \therefore & \neg(p \wedge q) \Rightarrow r \end{array}$$

in propositional form.

(d) Complete the truth table below for the propositional form of the argument.

p	q	r	
1	1	1	
1	1	0	
1	0	1	
1	0	0	
0	1	1	
0	1	0	
0	0	1	
0	0	0	

(e) Is the argument valid or invalid ? Why ?

(2) Consider the expression P defined by

p	q	r	s	P
1	1	1	1	1
1	1	1	0	1
1	1	0	1	1
1	1	0	0	0
1	0	1	1	1
1	0	1	0	0
1	0	0	1	1
1	0	0	0	1
0	1	1	1	1
0	1	1	0	0
0	1	0	1	0
0	1	0	0	1
0	0	1	1	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

(a) Write a disjunctive normal form for P.

(b) Write a conjunctive normal form for P.

(c) Write an optimized disjunctive normal form for P

by completing the Karnaugh map

(3) Consider the model

Objects	True Propositions
a	$P(a)$ $R(a)$ $S(a)$
b	$Q(b)$ $S(b)$
c	$P(c)$ $Q(c)$ $S(c)$
d	$P(d)$ $Q(d)$

For each of the following propositions determine its truth value in the model. **Justify your answer.**

(a) $\forall x\ \Big(\big(P(x) \vee Q(x)\big) \Rightarrow \big(R(x) \vee S(x)\big)\Big)$

(b) $\forall x\ \Big(\big(P(x) \wedge Q(x)\big) \Rightarrow \neg R(x)\Big)$

(c) $\exists x\ \big(S(x) \Rightarrow \neg R(x)\big)$

(d) $\forall x\ \big(P(x) \Rightarrow Q(x)\big)$

(e) $\Big(\forall x\ \big(P(x) \vee Q(x)\big)\Big) \Rightarrow \Big(\exists x\ \big(R(x) \wedge S(x)\big)\Big)$

(4) Test

$$\begin{array}{rl} & \forall x \ \big(P(x) \vee Q(x)\big) \\ & \exists x \ \big(P(x) \Rightarrow R(x)\big) \\ & \exists x \ \big(Q(x) \Rightarrow R(x)\big) \\ \therefore & \exists x \ R(x) \end{array}$$

for validity. **Clearly** indicate the contradiction or give a model which is a counterexample.

(5) (a) Give the formal definition of $A \subseteq B$ using logical symbols.

(b) By negating the previous definition give the formal definition of $A \nsubseteq B$.

(c) Give the contrapositive of the statement
If $A \backslash B = \varnothing$ then $A \subseteq B$.

(d) Using the above, or otherwise, prove the statement
If $A \backslash B = \varnothing$ then $A \subseteq B$.

(6) Prove that for all sets A, B

$$(A\backslash B) \cup (B\backslash A) = (A \cup B)\backslash(A \cap B).$$

using either a chain of equalities from the 12 laws or using the element method.

(7) A group of eight people attend the movies together

(a) If two of the eight insist on sitting together. In how many ways can the eight be seated in a row ? **Justify your answer.**

(b) Suppose instead, that two of the eight do not get along and won't sit together. In how many ways can the eight be seated in a row ? **Justify your answer.**

(c) Suppose the 8 people are made up of 4 women and 4 men. In how many ways can the eight be seated in a row such that sexes alternate ? **Justify your answer.**

(8) (a) Two drugs are to be tested on 60 lab mice. Each mouse recieves either one drug or acts as a control. Drug A is to be tested on 22 of the mice. Drug B is to be tested on another 22 of the mice. The remaining 16 mice are to act as a control group. How many different ways can the tests be performed ? **Justify your answer.**

(b) A bakery produces 6 different kinds of pastry.

(i) How many different selections of 13 pastries are there ? **Justify your answer.**

(ii) How many different selections of 13 pastries are possible if at least 3 of them must be eclairs ? **Justify your answer.**

(9) Prove by (regular) induction that:

For any real number $r \neq 1$ and integer $n \geq 0$

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}.$$

Clearly indicate the *base case*, what must be proved for the *inductive step*, and the *inductive hypothesis*.

(10) (a) Give the definition of a *relation R from X to Y*.

(b) Define what it means that a relation R from X to Y is a *function*.

(c) Define what it means for a function $f : X \to Y$ to be *one-to-one.*

(d) Define what it means for a function $f : X \to Y$ to be *onto*.

(e) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 3x^2$. Is f one-to-one, onto, or a bijection ? If it is a bijection give the inverse. **Justify your answer.**

(f) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 2$. Is f one-to-one, onto, or a bijection ? If it is a bijection give the inverse. **Justify your answer.**

(11) (a) Let R be an equivalence relation on X. Give the definitions of $[x]$, the *equivalence class* of an element $x \in X$.

(b) Let X be the set of 3 digit integers, i.e. $\{100, ..., 999\}$. Define R on X by xRy if the sum of the digits in x is the same as the sum of the digits in y.

(i) Show R is an equivalence relation on X.

(ii) Find the equivalence class $[121]$.

(c) Let R be an equivalence relation on X. Prove:

For all $x, y \in X$, if $[x] \cap [y] \neq \varnothing$ then $[x] = [y]$.

(12) (a) Give the definition of a *partial order* $\preceq$ on X. When is a *partial order* a *total order* ?

(b) Give an example of a *partial order* that is not a *total order*. Justify your answer.

(c) Let R be a relation on $\mathbb{R}$ defined by

$$xRy \text{ if } x^2 \leq y^2.$$

Is R a partial order ? Either prove it is, or show what property fails.