

MATH 361K EXAM 3

Name: _____
UT EID: _____

INSTRUCTIONS

- Please put your name and UT EID in the space provided.
- There are 9 questions each worth 10 points.
- You have 3 hours to complete the test.
- Please write your working and solutions on the test paper. You may use the back of the pages.
- Calculators are not allowed.
- Good luck ! It has been a pleasure.

FOR INSTRUCTOR'S USE

Question 1	_____
Question 2	_____
Question 3	_____
Question 4	_____
Question 5	_____
Question 6	_____
Question 7	_____
Question 8	_____
Question 9	_____
Total	_____

Problem 1

1. [3 Points] State the Nested Interval Property for \mathbb{R} . Clearly identify all the hypotheses.

2. [7 Points] Use the Nested Interval property to prove that the interval $[0, 1]$ is uncountable.

Hint: Proceed by contradiction.

Problem 2

1. [3 Points] Define what it means for $L \in \mathbb{R}$ to be the limit of the sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$.

2. [7 Points] Using the definition of the limit prove

Theorem. *Let $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $(b_n)_{n=1}^{\infty} \subset \mathbb{R}$ be convergent sequences. Then $(a_n \cdot b_n)_{n=1}^{\infty}$ is a convergent sequence and*

$$\lim_{n \rightarrow \infty} (a_n \cdot b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \cdot \left(\lim_{n \rightarrow \infty} b_n \right).$$

Problem 3

1. [2 Points] Define what it means for a sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ to be a Cauchy sequence.

2. [4 Points] Prove that any Cauchy sequence is bounded.

3. [4 Points] Prove

Theorem. *If $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ is a Cauchy sequence with a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ then $(a_n)_{n=1}^{\infty}$ is convergent and*

$$\lim_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} a_{n_k}.$$

Combining these two theorems and the Bolzano-Weierstraß theorem we can conclude that any Cauchy sequence in \mathbb{R} is convergent.

Problem 5

1. [2 Points] Define what it means for an infinite series $\sum_{n=1}^{\infty} a_n$ to converge absolutely.

2. [6 Points] Prove:

Theorem. Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with $a_n \neq 0$. If

$$\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

You may use the geometric definition of lim sup without proof.

3. [2 Points] Give an example of a convergent infinite series $\sum_{n=1}^{\infty} a_n$ with

$$\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

Problem 6

1. [2 Points] Let $A \subseteq \mathbb{R}$. Define what it means for $c \in \mathbb{R}$ to be a limit point of A .

2. [4 Points] Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$, and $c \in \mathbb{R}$ a limit point of A . Give the ϵ and δ definition of what it means for $L \in \mathbb{R}$ to be the limit of f at c .

3. [4 Points] Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$, and $c \in A$. Give the ϵ and δ definition of what it means for f to be continuous at c .

Problem 7

1. [2 Points] Let $f : (a, b) \rightarrow \mathbb{R}$ and $c \in (a, b)$. Define what it means for f to be differentiable at c .

2. [6 Points] Prove:

Theorem. Let $f : (a, b) \rightarrow \mathbb{R}$ and $c \in (a, b)$. f is differentiable at c if and only if there exists a function $\phi : (a, b) \rightarrow \mathbb{R}$ which is continuous at c such that

$$f(x) = f(c) + (x - c)\phi(x).$$

Moreover, we have $f'(c) = \phi(c)$.

3. [2 Points] Let $f : (a, b) \rightarrow \mathbb{R}$ and $c \in (a, b)$. Prove that if f is differentiable at c then f is continuous at c .

Problem 8

1. [6 Points] Prove

Theorem. *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, f is differentiable on (a, b) , and $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

2. [4 Points] Prove

Theorem. *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, f is differentiable on (a, b) , then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 9

1. [2 Points] Let $f : [a, b] \rightarrow \mathbb{R}$. Define what it means for the number I to be the *Riemann integral* of f over $[a, b]$.

2. [2 Points] Let $f : [a, b] \rightarrow \mathbb{R}$. Define $\overline{S}(f, \pi)$, the *upper Riemann sum* of f over the partition π , and $\underline{S}(f, \pi)$, the *lower Riemann sum* of f over the partition π .

3. [6 Points] Prove:

Theorem. *Let $f : [a, b] \rightarrow \mathbb{R}$. f is Riemann integrable if for every $\epsilon > 0$ there exists a partition π of $[a, b]$ such that*

$$\overline{S}(f, \pi) - \underline{S}(f, \pi) < \epsilon.$$