Homework 1 for M361K

- Please label your homework clearly with your name.
- Homework must be neatly written and must be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Tuesday January 31st

(1) Prove that $u = \sup S$ if and only if u is an upper bound for S that satisfies that for every $u' < u$ there exists $s \in S$ with $u' < s$.

Hint: Since this is an if and only if you need to prove the two statements:

(a) if $u = \sup S$ then u is an upper bound for S that satisfies that for every $u' < u$ there exists $s \in S$ with $u' < s$.

(b) if $u \in \mathbb{R}$ an upper bound for S satisfies that for every $u' < u$ there exists $s \in S$ with $u' < s$ then $u = \sup S$.

In each case the key is to work with the definition of the supremum and the definition of an upper bound for S.

(2) Prove that if S is a non-empty subset of $\mathbb{R}$ that is bounded from below then

$$\inf S = -\sup -S$$

where $-S = \{-s : s \in S\}$.

(3) Let S be a non-empty subset of $\mathbb{R}$. Prove that if there exists $u \in S$ with u an upper bound for S then $\sup S = u$.

(4) Let S be a bounded set in $\mathbb{R}$ and let S_0 be a non-empty subset of S. Prove that

$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S.$$

(5) Let

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Prove that $\inf S = 0$.

Hint: use our corollary to the Archimedean property.

(6) Give an example of:

(a) a nested sequence of bounded open intervals in $\mathbb{R}$ whose intersection is empty.

(b) a nested sequence of closed intervals in $\mathbb{R}$ whose intersection is empty.

(c) a nested sequence of closed and bounded intervals in $\mathbb{Q}$ whose intersection is empty.