Homework 4 for M361K

- Please label your homework clearly with your name.
- Homework must be neatly written and must be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Tuesday February 21st

(1) Prove:

Theorem. *If* (a_n) *is a sequence of real numbers with* $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

This lets us drop *positivity* as an assumption in many of the test theorems.

(2) Prove the following "improved" version of Theorem 3.2.11 (the Ratio Test):

Theorem. *Let* (a_n) *be a sequence of real numbers. If*

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

then the sequence (a_n) *converges and* $\lim_{n\to\infty} a_n = 0$.

You may wish to look at the content of Theorem 3.2.7 (The Squeeze Theorem).

(3) Find a sequence (a_n) of positive real numbers such that

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$$

but such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \text{ does not exist.}$$

(4) Prove the following counterpart

Theorem. *Let* (a_n) *be a sequence of real numbers. Suppose*

$$\liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

then the sequence (a_n) *diverges.*

You may use Theorem 3.2.9. In its contraspositive form it says that if $(|a_n|)$ diverges then so does (a_n).

(5) Prove the Root Test:

Theorem. *Let* (a_n) *be a sequence of real numbers. If*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$$

then (a_n) *converges and* $\lim_{n\to\infty} a_n = 0$.

(6) Let $|c| < 1$. Consider the sequence (a_n) given by $a_n = n^3 c^n$. Decide whether the sequence converges or not and justify your answer.

You may use all of our algebra of limit results.