

HOMEWORK 6 FOR M361K

- Please label your homework clearly with your name.
- Homework must be neatly written and must be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE TUESDAY THURSDAY MARCH 29TH

- (1) Prove the *Cauchy Condensation Criterion*:

Theorem. If (a_n) is a sequence in \mathbb{R} such that $a_{n+1} \leq a_n$ and $a_n > 0$ then

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converges}$$

Hint: The idea is to group the terms into blocks and approximate as we did for the p-test or harmonic series. In particular you should show

$$\begin{aligned} \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n}) &\leq s_{2^n} \\ s_{2^n} &\leq (a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1} a_{2^{n-1}}) + a_{2^n} \end{aligned}$$

- (2) Use the Cauchy Condensation Criterion to give an alternative proof of

Theorem. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

- (3) Use the Cauchy Condensation Criterion to show that

- (a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges
- (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^c}$ converges for $c > 1$.

- (4) Consider the series

$$\sum_{n=1}^{\infty} 2^{(-1)^n - n}$$

Show that the Ratio test fails since

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{8} \quad \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2$$

but show that the series converges by applying the root test.

Hint: In fact you can rely too much on these tests. This series is trivially convergent by direct comparison with $\sum_{n=1}^{\infty} 2^{1-n}$.