

HOMWORK 8 FOR M361K

- Please label your homework clearly with your name.
- Homework must be neatly written and must be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

DUE TUESDAY APRIL 18TH

- (1) Prove from the definition that if $f : A \rightarrow \mathbb{R}$ and $c \in A$ is not a cluster point of A then f is continuous at c .

Remark: We call such points *isolated*. Many of our theorems assume that $c \in A$ is a cluster point of A since then the proof proceeds without having to treat the case of isolated points separately. If $c \in A$ is not a cluster point we cannot speak about limits.

- (2) Prove the following theorem:

Theorem (Sequential Definition of Continuity). *Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$, and $c \in A$. The function f is continuous at c if and only if for all sequences (a_n) in A with $\lim_{n \rightarrow \infty} a_n = c$ we have*

$$\lim_{n \rightarrow \infty} f(a_n) = f(c).$$

Remark: This is very similar to our sequential definition of limit except that the restriction $a_n \neq c$ is dropped. We don't need to restrict c to be a cluster point of A since continuity is automatic for isolated points.

- (3) Let $f : \mathbb{R} \rightarrow \mathbb{R}$, and $g : \mathbb{R} \rightarrow \mathbb{R}$. Find examples to show
- (a) if $f + g$ is continuous at c then neither f nor g need to be continuous at c .
 - (b) if $g \circ f$ is continuous at c then f need not be continuous at c and g need not be continuous at $f(c)$.
- Hint: though it is possible to find a single example where f and g are both discontinuous it is easier to give two examples, one where f is discontinuous and one where g is discontinuous.
- (4) Suppose $f : A \rightarrow \mathbb{R}$ is uniformly continuous. If $(a_n) \subseteq A$ is Cauchy show that $(f(a_n))$ is a Cauchy sequence in \mathbb{R} .

- (5) Find an example of a continuous function $f : (0, 1) \rightarrow \mathbb{R}$ such that the sequence $(f(\frac{1}{n}))$ is not Cauchy.

Remark: The easiest way to do this is with a function such that $\lim_{x \rightarrow 0} f(x) = +\infty$ though you can do it with a rapidly oscillating bounded function too.

- (6) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $g(x + y) = g(x)g(y)$. Show that if g is continuous at 0 then g is continuous at all points of \mathbb{R} .

Remark: Later we will show that if this function is continuous at $x = 0$ then this function must be $g(x) = Ce^x$.

Hint: If (x_n) converges to x then $x_n - x$ converges to 0. We can write $x_n = (x_n - x) + x \dots$