Homework 9 for M361K

- Please label your homework clearly with your name.
- Homework must be neatly written and must be stapled.
- Feel free to discuss your solutions with other students but try to solve the problems by yourself first.

Due Tuesday April 25

(1) A function f is uniformly continuous on the *open* interval (a, b) if and only if it can be extended to a continuous function on the *closed* interval $[a, b]$.

Remark: This theorem can be considerably extended. If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ is uniformly continuous on A then f can be extended to a continuous function $\hat{f} : \overline{A} \to \mathbb{R}$ where $\overline{A}$ is A together with all its cluster points (this is called the closure of A). Moreover this extension is unique.

(2) Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $g(1) = C > 0$ and $g(x + y) = g(x)g(y)$. Show that g is defined on $\mathbb{Q}$ by

$$g\Big(\frac{m}{n}\Big) = C^{\frac{m}{n}}.$$

Remark: Since g is uniformly continuous on every $[a, b]$ and $\overline{\mathbb{Q} \cap [a, b]} = [a, b]$ the more powerful version of the previous theorem then gives that $f(x) = C^x$. Another way of saying this is *the only continuous functions that turn addition into multiplication are the exponential functions.*

(3) Let $f : [0, 1] \to \mathbb{R}$ be a function that takes on every one of its values exactly twice. Show that f cannot be continuous on $[0, 1]$.

Hint: Consider the two points where f achieves its supremum. Show that if f is continuous then these must be 0 and 1 respectively. Now consider the points where f achieves its infimum.

(4) Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$ be an increasing function. Let $c \in I$ with c not an endpoint. Show that f is continuous at c if and only if there exists a sequence $(x_n) \subset I$ with $x_n < c$ for n odd, $x_n > c$ for n even, with $\lim_{n\to\infty} x_n = c$, such that $\lim_{n\to\infty} f(x_n) = f(c)$.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Using the definition show that f is differentiable at $x = 0$. Is f differentiable anywhere else ?

(6) Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 \sin(x^{-2})$ for $x \neq 0$ and $f(0) = 0$ is differentiable everywhere but that the derivative is not continuous at $x = 0$.

Hint: By the rules for differentiation $f(x)$ is obviously differentiable everywhere except $x = 0$. For $x = 0$ you need to use the definition and the squeeze theorem.