Math 427K Final Exam

Name:	____________________
UT EID:	____________________

Instructions

- Please put your name and UT EID in the space provided.
- There are 10 questions each worth 10 points. There should be a total of 12 pages including this one and the Laplace table.
- You have 3 hours to complete the test.
- Please write your working and solutions on the test paper. You may use the back of the pages.
- Calculators are not allowed.

Method of Undetermined Coefficients Table

$g(t)$	$Y(t)$
$P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$	$t^s(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_0)$
$P_n(t)e^{\alpha t}$	$t^s(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_0)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin(\beta t)\\ \cos(\beta t)\end{cases}$	$t^s\big[(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_0)e^{\alpha t}\cos(\beta t) + (B_n t^n + B_{n-1}t^{n-1} + \cdots + B_0)e^{\alpha t}\sin(\beta t)\big]$

For Instructor's Use

Question 1	______
Question 2	______
Question 3	______
Question 4	______
Question 5	______
Question 6	______
Question 7	______
Question 8	______
Question 9	______
Question 10	______
Total	______

Question 1

Find the solution of the initial value problem

$$y' + t\,y = t, \qquad y(0) = 2.$$

Question 2 [10 Points]

Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 - 1}{4 + 2y}, \qquad y(1) = -1$$

explicitly.

Question 3 [10 Points]

Find the general solution of the ordinary differential equation

$$(3x^2y + 2xy + y^3)\,dx + (x^2 + y^2)\,dy$$

You do not need to find the solution explicitly.

Question 4 [10 Points]

Consider the autonomous ordinary differential equation

$$\frac{dy}{dt} = f(y) \quad \text{where} \quad f(y) = 2(y-2)^3(y+1)^2(y+3).$$

1. [4 Points] Sketch $f(y)$ vs y.

2. [4 Points] Find the equilibrium solutions and classify each according to its stability.

3. [2 Points] Sketch $y(t)$ vs t.

Question 5 [10 Points]

1. Find the general solution of the inhomogeneous second order ordinary differential equation
$$y'' - 2y' - 3y = (-8t + 6)e^{-t}.$$

2. Find the *complementary function* and a *suitable form for a particular solution* of
$$y'' - 2y' + 2y = e^{t} + e^{t}\cos(t) + \sin(t).$$
You do not need to find the undetermined coefficients.

Question 6 [10 Points]

Find the general solution of the ordinary differential equation

$$x^2 y'' + 5xy + 4y = 0$$

given that

$$y_1(x) = x^{-2}$$

is a solution.

Question 7 [10 Points]

Find the general solution of the ordinary differential equation

$$y'' + xy' + 2y = 0$$

about the point $x_0 = 0$ by using power series methods. You should find the recurrence relation and formula for the general term if possible.

Question 8 [10 Points]

Solve the initial value problem

$$y'' + 2y' + 2y = g(t), \qquad y(0) = 0,\, y'(0) = 0$$

where

$$g(t) = \begin{cases} 5\sin(t) & 0 \leq t < \pi \\ 0 & \pi \leq t \end{cases}$$

using the partial fractions expansion

$$\frac{5}{(s^2+1)(s^2+2s+2)} = \frac{2s+3}{s^2+2s+2} - \frac{2s-1}{s^2+1}$$

Question 9 [10 Points]

Determine the eigenvalues and eigenfunctions for the boundary value problem

$$y'' + \lambda y = 0, \qquad y(0) = 0,\ y'(\pi) = 0.$$

You may assume that there are no negative eigenvalues.

Question 10 [10 Points]

Suppose the temperature $u(x,t)$ of a metal bar of length 10 cm satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

Suppose that both ends of the bar are held at $0°C$ and that the initial temperature distribution is

$$f(x) = \begin{cases} 4x & 0 \leq x \leq 5 \\ 40 - 4x & 5 \leq x \leq 10 \end{cases}$$

1. [2 Points] Give the basic solutions of the boundary value problem.

2. [8 Points] Find the solution $u(x,t)$ which satisfies the initial condition. You may leave your coefficients in terms of cos and sin.