Symmetry in Fourier Analysis — From Heisenberg to Stein–Weiss

objective Develop a paradigm for Analysis on Lie Groups

Embedded Symmetry & Encoding Information

Construct a toolbox of calculations

Estimates are easy — sharp constants are hard

Symmetry is intrinsic to the framework for analysis

— Patterns of Symmetry

Feynman Philosophy: you have to take things apart before you can understand how they work

Arnold: deep understanding comes when geometric insight is coupled with analytic computation

Tools of Fourier Analysis

Fourier transform — convolution — Young's inequality

Riesz potentials — Hardy–Littlewood–Sobolev inequality

Sobolev embedding — Stein–Weiss inequality — Laplacian

Focus functional $\boxed{\int_{M \times M} f(w)\, K[d^2(w,w')]\, f(w')\, dw\, dw'}$

structural symmetry

— line of duality $f \in L^p(M)$, $1 < p \le 2$ $\quad M = M_n \times \Sigma_m$

coupled metrics

$$d_M^2 = d_M^2 + d_\Sigma^2 = \Lambda[d_M^2, d_\Sigma^2]$$

In determing this functional — related to convolution and a group action on the manifold — break this symmetry to obtain the right estimate — Kunze – Stein Phenomena on $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$, Lorentz group

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importance of fundamental solutions & Laplacian

Drive for sharp constants — encode information

Hardy – Littlewood – Sobolev inequality
$\rightsquigarrow$ Moser – Trudinger inequality

Focus — Stein-Weiss integral with mixed homogeneity

(1) $\int_{M\times M} f(v)\,|y|^{-\alpha}\,|v-v'|^{-\lambda}\,|y'|^{-\alpha}\,g(v')\,dv\,dv' \lesssim A\,\|f\|_{L^p(M)}\,\|g\|_{L^p(M)}$

$v=(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$, $0<\lambda<n+m$, $0<\alpha<m/p'$, $\lambda = \frac{2n}{p'} + 2\left(\frac{m}{p'}-\alpha\right)$

1st example with sharp constants — Heisenberg group for Sobolev embedding with sharp constants for radial functions

~~Also~~, Provides example of Stein-Weiss integral on Heisenberg group with mixed homogeneity with

(2) $\int_{\mathcal{H}_n\times\mathcal{H}_n} f(|z|,t)\,|z|^{\beta}\,\rho(w,w')^{-2n}\,|z'|^{\beta}\,f(|z'|,t')\,dw\,dw'$ $\qquad w=(z,t)$

Hyperbolic Equivalence to (1)

(3) $\int_{N\times N} F(w,\xi)\left[d^2(w,w')+\tfrac14|\xi-\eta|^2\right]^{-\lambda/2} G(w',\eta)\,d\nu\,d\nu'\,d\xi\,d\eta$

$\lesssim C_p\,\|F\|_{L^p(N)}\,\|G\|_{L^p(N)}$ $\qquad N=\mathbb{H}^{n+1}\times S^{m-1}$

Why 2 is natural — and can not be touched by the methods of Stein-Weiss or Hardy-Littlewood — they rely on applying Young's inequality — see the radial lemma for Sobolev embedding — $\left[\int_{\mathbb{R}^4} |y|\, |f(y,\hat{x}_0)|^5 \,dx\right]^{1/5} \leq C \,\|\nabla f\|_{L^2(\mathbb{R}^4)}$
[see Strauss, Ni, Tao]

Extends to $\left[\int |y|^\beta |f(y,\hat{x}_0)|^q \,dx\right]^{2/q} \leq C \int_{\mathbb{R}^n} |(-\Delta)^{\delta/2} f|^2 \,dx$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} g(x)\, |x'|^\beta\, |x-y|^{-\lambda}\, |y'|^\beta g(y)\,dx\,dy \leq C\left[\|g\|_{L^p(\mathbb{R}^n)}\right]^2$$

$\simeq$ estimate on hyperbolic space $\qquad x = (x';\hat{x}) \in \mathbb{R}^{n-1}\times\mathbb{R}$

These examples demonstrate the importance of study for both embedding & potentials on hyperbolic space, and embedded symmetry on the Heisenberg group — Eli Stein's lecture for the ICM in 1970 at Nice outlined the importance of the two groups — $SL(2;R)$ and the Heisenberg group for moving analysis beyond the Euclidean framework.

H^n = n-dimensional hyperbolic space

$\mathcal{H}_n$ = Heisenberg group = boundary of the Siegel upper half space in $\mathbb{C}^{n+1}$

{ negative curvature
exponential growth of balls
non-homogeneous dilations

the simplest domain that breaks the uniform homogeneity of space

$\mathbb{R}^n$ = Euclidean manifold

Two principles

(I) intrinsic character of the Heisenberg group makes it the natural playing field to explore the laws of symmetry and the interplay between analysis & geometry on a manifold

Heisenberg group = manifold $\mathbb{R}^{2n} \times \mathbb{R}^m \ni v = (x, y)$

where the group action on $\mathbb{R}^m$ is coupled with a bilinear symplectic form on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ so that the homogeneous dimension of the manifold is $N = 2n + 2m$

(II) Fundamental Identities, Formulas & Inequalities

are essential for developing analysis on a geometric manifold

— explicit, elegant, influential

(a) encode information

(b) most basic computational recipe

(c) lead to new mathematical structure

Basic Formulas for the Heisenberg Group

(1) $$\varphi(0) = a_n \int_{\mathcal{H}_n} \varphi(w) \left[-\Delta_{\mathcal{H}}\, \rho^{-(N-2)} \right] dm$$

$$= 4^n a_n b_n \int_{\mathbb{R}^{n+2}} \tilde{\varphi}(x) \left[-\Delta_{n+2} |x|^{-n} \right] dx$$

$$\tilde{\varphi}(x) = \tilde{\varphi}(|z|, t) = \int_{S^{2n-1}} \varphi(w)\, d\sigma$$

$|x| = \sqrt{|z|^2 + t^2}$ $\qquad x = (x', x_0),\ x' \in \mathbb{R}^{n+1},\ x_0 = \mathbb{R}$ $\qquad |z|^4 \rightsquigarrow |z|^2 = |x'|^2$

Acting on radial functions

$$\Delta_{\mathcal{H}} = \Delta_{2n} + 4|z|^2 \frac{\partial^2}{\partial t^2}$$

(2) $f(z,t)$ radial in the z variable

$$\left[\|\nabla_{\mathcal{H}} f\|_{L^2(\mathcal{H}_n)}\right]^2 = \int_{\mathcal{H}_n} f(-\Delta_{\mathcal{H}} f)\, dm = c\int_{\mathbb{R}^{n+2}} f(-\Delta_{n+2} f)\, dx$$
$$= c\left[\|\nabla f\|_{L^2(\mathbb{R}^{n+2})}\right]^2$$

Two embedding estimates

(a) $\displaystyle\int_{\mathbb{R}^{n+2}} |\nabla f|^2\, dx \geq C_r \left(\int_{\mathbb{R}^{n+2}} |f|^r\, dx\right)^{2/r} \qquad r = 2(n+2)/n$

(b) $\displaystyle\int_{\mathcal{H}_n} |\nabla_{\mathcal{H}} f|^2\, dm \geq B_q \left(\int_{\mathcal{H}_n} |f|^q\, dm\right)^{2/q}$

Different Extremals

$q = 2(n+1)/n \qquad r > q$

Here we use sharp Sobolev embedding on Euclidean space and sharp Sobolev embedding on $\mathbb{R}^n$ and sharp Sobolev embedding on the Heisenberg group — the latter was obtained by the author using transference to hyperbolic space (see author's paper on Grushin operators — appendix) or by mapping the problem from the Heisenberg group to the boundary of the complex ball in $\mathbb{C}^{n+1}$ where the operator is controlled by the Hardy-Littlewood-Sobolev inequality on the sphere S^{2n+1}

Flip the two estimates above —

(c) $\displaystyle\int_{\mathbb{R}^{n+2}} |\nabla f|^2\, dx \geq B_q \left(\int_{\mathcal{H}_n} |f|^q\, dm\right)^{2/q} = c\left[\int_{\mathbb{R}^{n+2}} |x'|^{-1} |f(x)|^q\, dx\right]^{2/q}$

this leads to a Stein-Weiss inequality on $\mathbb{R}^{n+2}$ with mixed homogeneity.

(d) $\displaystyle\int_{\mathcal{H}_n} |\nabla_{\mathcal{H}} f|^2\, dm \geq C_r \left(\int_{\mathbb{R}^{n+2}} |f|^r\, dx\right)^{2/r} = c\left[\int_{\mathcal{H}_n} |z|^2\, |f|^r\, dm\right]^{2/r}$

this leads to a Stein-Weiss inequality on $\mathcal{H}_n$ with positive weights