Mean Field games and related topics

In this series of lectures, we discuss a new type of mathematical models, with applications to Economics, Finance and Biology, derived from a general approach that we call a mean field approach by analogy with Statistical Mechanics and Physics. Roughly speaking, this approach consists in considering situations that involve a large number of rational players whose decisions effectively depend on limited informations. Here, rational means that each player chooses its strategy in an optimal way (in the sense of optimal control).Such an approach contains as a particular case the classical situations in Physics where one considers systems composed of a very large number of particles with « weak » (limited) interactions and one dervives Nonlinear Partial Differential Equations that are « mean field » or self-consistent equations.

We present three examples of such an approach, taken from a series of works in collaboration with Jean-Michel Lasry :

- the first one concerns the formation of volatility in financial markets and leads to a nonlinear differential equation in infinite dimensions that, as we show, is well-posed and can be solved semi-explicitely,
- the second one concerns the dynamical formation of prices and we are led in the simplest idealized case to a new free-boundary problem that we solve (introducing an effective algorithm to compute the solution)
- the third example corresponds to the main topic of these lectures namely what we call mean-field games.

On this last topic, we begin by recalling the notion of Nash equilibia for N-players stochastic differential games and briefly discuss existence and non-uniqueness issues for these equilibria. We next explain formally the type of models one obtains as the number of players goes to infinity and present the known mathematical results on this limit. We then discuss the mathematical structure of this new class of Nonlinear Partial Differential Equations and show that it contains as particular examples the Hartree models in Quantum Mechanics or some Fluid Mechanics models such as the Euler equations. We also mention many open problems together with a variety of directions for future work that need to be addressed.

(During the lectures, copies of a survey paper to be published in march 07 in the Japanes Journal of Mathematics will be distributed, with a list of references)