QUIVERS AND LATTICES.

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We will discuss two classification results in quite different areas which turn out to have the same answer. This note is an slightly expanded version of the talk given in the GRASP seminar of March 2005.

1. Lattices

A positive definite lattice L is a free abelian group with a symmetric bilinear form $(\cdot,\cdot)\colon L \times L \to \mathbb{Z}$ such that $N(v) = (v,v) > 0$ for all $v \in L - \{0\}$. We call $N(v)$ the *norm* of $v, (v \in L)$. Let V be the real vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$ which has a positive definite inner product induced from $(\cdot,\cdot)$, indeed we may think of L as simply a discrete subgroup of the Euclidean vector space V on which the inner product restricts to be integer valued.

Our first problem is to classify positive definite lattices generated by norm 2 vectors, thus from now on we assume that L is such a lattice. Let

$$R = \{\alpha \in L : N(\alpha) = 2\}$$

be the set of *roots* of L. For each root α we have a hyperplane

$$H_\alpha = \{x \in V : (x,\alpha) = 0\},$$

(so of course $H_\alpha = H_{-\alpha}$). Finally, for each α in R let s_α denote the reflection in H_α. Since

$$s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha = x - (x,\alpha)\alpha,$$

we see that s_α is an automorphism of the lattice. The subgroup W of $Aut(L)$ generated by the s_α is known as the Weyl group. Since it preserves the norm 2 vectors, it is clearly a finite group.

The complement of $\bigcup_{\alpha\in R} H_\alpha$ is an open subset $\mathcal{U}$ of V consisting of a number of connected components known as *chambers*. Given a chamber $\mathcal{C}$ we say that H_α is a *wall* of $\mathcal{C}$ if the closure of $\mathcal{C}$ intersects H_α in an open subset of H_α. If $\mathcal{C}, \mathcal{C}'$ are two chambers separated by a hyperplane H_α then since s_α is a lattice automorphism, we must have $s_\alpha(\mathcal{C}) = \mathcal{C}'$, and hence it follows that $Aut(L)$ acts transitively on the set of chambers, and so any two are "isomorphic". Fix one, $\mathcal{C}$ say, and let Δ be the set of roots corresponding to the walls of $\mathcal{C}$ such that $\mathcal{C}$ and α lie on the same side of H_α. These are known as the *simple roots* corresponding to $\mathcal{C}$, and

$$\mathcal{C} = \bigcap_{\alpha\in\Delta} \{x \in V : (\alpha, x) > 0\}$$

.

Lemma 1.1. *If $\alpha, \beta \in \Delta$ are such that $\alpha \neq \pm\beta$, then $(\alpha,\beta) \in \{0,-1\}$.*

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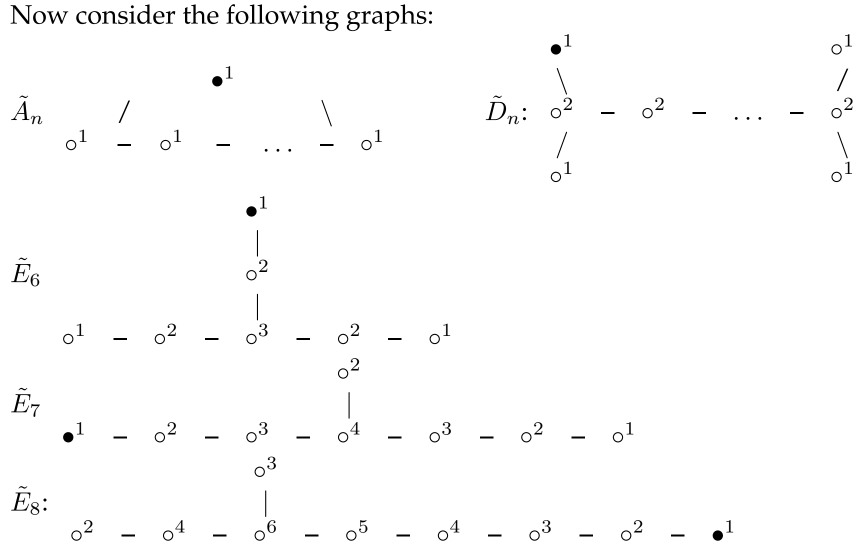
Proof. By the Cauchy-Schwarz inequality it is clear that $(\alpha, \beta) \in \{-1, 0, 1\}$ and so we only need to show that it is not 1. Suppose the opposite: then $s_\alpha(\beta) = \beta - \alpha$ is a root of L, and so since $H_{\beta-\alpha}$ separates α and β they cannot be wall of $\mathcal{C}$, which is a contradiction. $\square$

Using the lemma we build a graph Γ_L as follows: we let the nodes of Γ_L correspond to the elements of Δ and put an edge between the nodes corresponding to α and β if $(\alpha, \beta) = -1$. We call this the *Dynkin diagram* associated to L

Now consider the following graphs:



Suppose that one of these graphs occured as a subgraph of Γ_L. Then if the nodes correspond to simple roots $\alpha_1, \alpha_2, \dots, \alpha_r$ say, we can form the vector $u = \sum_{i=1}^r m_i \alpha_i$ where the m_i are the positive integers labeling nodes in the above graphs. Now $u \neq 0$ since it is a positive linear combination of elements of Δ, and so for any $x \in \mathcal{C}$ we have $(u, x) > 0$. However $(u, \alpha_i) = 2m_i - \sum_{j-i} m_j = 0$, and hence $(u, u) = 0$, and this contradicts the fact that $(\cdot, \cdot)$ is positive definite. Thus the graph Γ_L does not contain any of the graphs $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$ as a subgraph. It is now a simple exercise to see that the connected components of Γ_L are graphs of the form A_n, D_n, E_n where the graph X_n is obtained from $\tilde{X}_n$ by removing the node drawn as a solid dot: Fix a component Y and observe that Y must be a tree, since it cannot contain a graph of type $\tilde{A}_n$. Then note that since Y cannot contain $\tilde{D}_4$ every node has at most 3 edges incident to it, and moreover since it cannot contain a graph of type $\tilde{D}_n$ for $n \geq 5$ there can be at most one node with 3 edges incident to it. Finally we can use the graphs $\tilde{E}_n$ to conclude that Y is of the form A_n, D_n or E_n as claimed.

It remains only to check that there actually are lattices corresponding to the diagrams A_n, D_n, E_n – in fact they are the following, where the positive definite form comes from the standard one on $\mathbb{R}^n$.

- A_n: $L = \{(a_1, a_2, \dots, a_{n+1}) \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} a_i = 0\}$.
- D_n: $L = \{(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} a_i \in 2\mathbb{Z}\}$.
- E_8: $L = \{(a_1, a_2, \dots, a_8) \in \mathbb{R}^8 : \sum_{i=1}^{8} a_i \in 2\mathbb{Z},$ and $\forall i, a_i \in \mathbb{Z}$, or $\forall i, a_i \in \frac{1}{2} + \mathbb{Z}\}$
- E_7: $L = \{(a_1, a_2, \dots, a_8) \in \mathbb{E}_8 : m_1 = m_2\}$.

- E_6: $L = \{(a_1, a_2, \dots, a_8) \in \mathbb{E}_8 : m_1 = m_2 = m_3\}$

To see that these correspond to the diagrams we just need to give a collection of simple roots:

- A_n: such a collection is given by $(\alpha_i)_{1\leq i\leq n}$ where if $(e_i)_{1\leq i\leq n+1}$ are the standard basis vectors of $\mathbb{R}^n$
$$\alpha_i = e_{i+1} - e_i.$$
- D_n: such a collection is given by the vectors $(\alpha_i)_{1\leq i\leq n}$ where for $i \leq n-1$ we put
$$\alpha_i = e_{i+1} - e_i$$
and we let $u_n = e_{n-1} + e_n$.
- E_n: For E_8 such a collection is given by $(\alpha_i)_{1\leq i\leq 8}$ where $\alpha_i = e_{i+1} - e_i$ for $1 \leq i \leq 7$ and
$$\alpha_8 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$$
For E_7 and E_6 we may simple take the roots $(\alpha_i)_{2\leq i\leq 8}$ and $(\alpha_i)_{3\leq i\leq 8}$ respectively.

An arbitrary positive definite lattice generated by norm 2 vectors is then isomorphic to an orthogonal direct sum of these lattices and we have obtained a complete classification:

Theorem 1.2. *Every positive definite lattice which is generated by its vectors of norm* 2 *is isomorphic to an orthogonal direct sum of lattices of type A_n, D_n, E_6, E_7 or E_8.*

Remark 1.3. To any lattice L we can associate a matrix C known as the Cartan matrix by picking a set of simple roots $\Delta = \{\alpha_i\}_{1\leq i\leq n}$ and setting
$$c_{ij} = (\alpha_i, \alpha_j) \in \{0, -1, 2\}.$$
Thus C is the matrix of the inner product with respect to the vectors in Δ. It is clear from our lemma that the data of the Cartan matrix is equivalent to the data of the graph Γ_L (indeed to any graph we can associate a Cartan matrix in an obvious way, and vice versa). One way of saying what we have done is that any graph Γ whose associated Cartan matrix $C_\Gamma = (c_{ij})_{1\leq i\leq k}$ satisfies
$$\mathbf{d}^t C \mathbf{d} > 0, \forall \mathbf{d} \in \mathbb{Z}^k_{\geq 0} - \{\mathbf{0}\}$$
is a union of Dynkin diagrams. We will use this fact later.

Remark 1.4. It is easy to check from the diagrams that the Cartan matrix corresponding to a union of Dynkin diagrams is nondegenerate, and so any set of simple roots is linearly independent. (Of course you can also see this directly from the explicit description given above.) A consequence of this is that every root can be written as a combination of simple roots $\alpha \in \Delta$ with coefficients either all nonnegative or all nonpositive: Let $\{\varpi_i\}$ be the basis of V dual to $\Delta = \{\alpha_i\}$. Then each ϖ_i is clearly in $\bar{\mathcal{C}}$ the closure of $\mathcal{C}$, and hence for any root α we must have $(\alpha, \varpi_i) \geq 0$ for all i or $(\alpha, \varpi_i) \leq 0$ for all i. But $\alpha = \sum(\alpha, \varpi_i)\alpha_i$ and so we are done. Thus given a set of simple roots we get a partition of R into *positive* and *negative* roots.

2. QUIVERS

A quiver Q is a direct graph, that is, Q consists of a set of nodes I and a set of directed edges H, where for an edge $h \in H$ we write $s(h)$ for the initial node of the edge, and $t(h)$ for the terminal node of the edge, *i.e* the edge is $s(h) \to t(h)$. We will assume that Q has no directed loops or multiple edges.

A representation (V, x) of Q over a field $\mathbf{k}$ is an I-graded vector space $V = \bigoplus_{i \in I} V_i$ and a collection of linear maps $x = (x_h)_{h \in H}$ where $x_h \in \mathrm{Hom}(V_{s(h)}, V_{t(h)})$. There is an obvious notion of morphism between representations of Q, and the resulting category is denoted $\mathcal{R}(Q)$. Now this category is in fact equivalent to the category of representations of an algebra known as the *path algebra*, $\mathbf{k}(Q)$ of Q. A *path* of length l in Q is a sequence of edges $[h_1, h_2, \ldots, h_l]$ such that $t(h_j) = s(h_{j+1}), (1 \leq j \leq l-1)$, if $l > 0$, and simply a node $[i]$ if $l = 0$. The algebra $\mathbf{k}(Q)$ is the $\mathbf{k}$-algebra with basis the set of paths in Q, and multiplication given by

$$[h_1, h_2, \ldots, h_k].[h'_1, h'_2, \ldots, h'_l] = \begin{cases} [h_1, h_2, \ldots, h_k, h'_1, \ldots h'_l] \text{ if } t(h_k) = s(h'_1), \\ 0 \text{ otherwise,} \end{cases}$$

where for $i \in I$ we set $s(i) = t(i) = i$ in order to define multiplication by paths of length 0. Since $\mathbf{k}(Q)$ is generated by the paths of length at most 1, we see immediately that the category $\mathcal{R}(Q)$ can be identified with $Rep(\mathbf{k}(Q))$. Given a representation (V, x) of Q its *dimension* $\dim(V)$ is the tuple $(\dim(V_i))_{i \in I} \in \mathbb{N}^I$. Given a dimension $\mathbf{d} = (d_i)$ and an I-graded vector space V of dimension $\mathbf{d}$ the isomorphism classes of representations of Q which have dimension $\mathbf{d}$ correspond to the orbits of the group $\prod_{i \in I} GL(V_i)$ on

$$E_V = \bigoplus_{h \in H} \mathrm{Hom}(V_{s(h)}, V_{t(h)}),$$

where the action is the obvious "conjugation".

We have a notion of simple objects and indecomposable objects in $\mathcal{R}(Q)$ – a simple representation is one which has no nonzero subrepresentations, while an indecomposable representation is one which cannot be written as the direct sum of two subrepresentations. If you are familiar with the theory of representations of finite groups over $\mathbb{C}$ you may know that in that situation the two notions coincide, however for representations of quivers this is almost never the case. Indeed it is easy to see that the isomorphism classes of simple representations of a quiver Q are labelled by the nodes of Q: let $S_i = (V_k)_{k \in I}$ where $V_k = 0$ if $k \neq i$ and $V_i = \mathbf{k}$, (thus the maps x_h are necessarily all zero). Any simple representation of Q is isomorphic to S_i for some $i \in I$.

Example 2.1. Consider the quiver with two nodes and one arrow joining them. Then the representation

$$\mathbf{k} \quad \to \quad \mathbf{k}$$

where the map is an isomorphism is indecomposable but not simple. It is not to hard to see that this is the only nonsimple indecomposable representation of this quiver. (In particular the classification of indecomposables is independent of the field $\mathbf{k}$.)

Thus in contrast to the case of finite group representations over $\mathbb{C}$, it is not the case that every representation is a sum of simple ones. However it is the case that every representation is a sum of indecomposable representations in an essentially unique way (this is known as the Krull-Schmidt theorem). Hence if we wish to understand the structure of representations of Q we are reduced to understanding the structure of indecomposable representations. Gabriel asked which quivers would have *finite representation type*, that is, finitely many indecomposable representations, and found the following beautiful answer.

Theorem 2.2. *For any field $\mathbf{k}$, a connected quiver Q has finite representation type precisely if it's underlying undirected graph is a simply laced Dynkin diagram. Moreover the indecomposables correspond via their dimension to a set of positive roots of the corresponding lattice.*

Proof. We give here only a partial explanation of the theorem in the case that $\mathbf{k}$ is algebraically closed, using an argument due to Tits. Suppose that Q has only finitely many indecomposable representations. Then by the Krull-Schmidt theorem, in a given dimension $\mathbf{d}$ there can only be finitely many isomorphism classes of representations. But as we saw earlier this means that if V is a vector space of dimension $\mathbf{d}$ then there are only finitely many orbits of the group $G_V = \prod_{i \in I} GL(V_i)$ on the vector space $E_V = \bigoplus_{h \in H} \mathrm{Hom}(V_{s(h)}, V_{t(h)})$. Now this means that necessarily one of these orbits will be open dense in E_V, and so in particular we can do a dimension count to see if there is a chance for this to be true. Now

$$\dim(G_V) = \sum_{i \in I} d_i^2, \qquad \dim(E_V) = \sum_{h \in H} d_{s(h)} d_{t(h)},$$

and if there is a dense orbit we must have $\dim(G_V) - \dim(E_V) \geq 0$. However the "diagonal" copy of $\mathbf{k}^*$ acts trivially on E_V, and so in fact we need

$$\dim(G_V) - \dim(E_V) \geq 1$$

Introduce a matrix $C = (c_{ij})_{i,j \in I}$ by setting $c_{ii} = 2$ and $c_{ij} = -1$ if there is an edge (in either direction) between i and j, and otherwise $c_{ij} = 0$. Thus C is the Cartan matrix of the underlying undirected graph associated to the quiver. Then the above inequality may be expressed as:

$$\mathbf{d} C \mathbf{d}^t > 0, \mathbf{d} \in \mathbb{N}^I, \mathbf{d} \neq 0,$$

(where we are thinking of $\mathbf{d}$ as a row vector). But now Remark 1.3 says exactly that the underlying graph of Q must be a Dynkin diagrams as claimed. Notice also that if $\mathbf{d}$ corresponds to a root of the associated lattice, and V has dimension $\mathbf{d}$ then in fact

$$\dim(G_V) - \dim(E_V) = 1,$$

and so we see that if $x \in E_V$ is in the dense orbit, and we make V into a representation of $\mathbf{k}(Q)$ using x, then $\mathrm{Hom}_{\mathbf{k}(Q)}(V, V)$ is one-dimensional. It is easy to see that this implies that (V, x) is indecomposable, and so we also see that there are indecomposables of this dimension (though from what has been said it is *not* clear that these are all the indecomposables)

□

Remark 2.3. The theorem in fact holds for any field, and in a beautiful paper Gelfand, Bernstein and Ponomarev show how to use operations called *reflection functors* to classify the indecomposables for a quiver whose underlying graph is a Dynkin diagram. Since these make sense for any field, the classification is independent of the field $\mathbf{k}$. This is in sharp contrast to the case of a finite group G where the characteristic of the field $\mathbf{k}$ changes the nature of the representations dramatically. Indeed already for the group $\mathbb{Z}/2\mathbb{Z}$ it is easy to see that if $\text{char}(\mathbf{k}) \neq 2$ then there are two irreducible representations and every representation is a direct sum or irreducible representations, while if $\mathbf{k}$ has characteristic 2 the trivial representation is the only irreducible, but it is not the case that every representation is trivial (if you don't already know this you should treat it as an exercise!)

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