1. Solutions

(1.5, 22) (a) Prove that \( \text{Tr}(AA^T) \) is the sum of the squares of all entries of \( A \).
(b) Prove that if \( \text{Tr}(AA^T) = 0 \), then \( A = 0 \).
(c) Prove that \( \text{Tr}(AB) = \text{Tr}(BA) \).

Proof. (a) Consider the \( i \)th diagonal entry of \( AA^T \). This consists of the dot product of the \( i \)th row with itself, which is precisely the sum of the squares of entries in the \( i \)th row. Since the trace is the sum of the diagonals, we see that \( \text{Tr}(AA^T) \) is the sum

\[
\sum_{i,j} A_{ij}^2.
\]

(b) Since each squared term in the sum above is positive, the sum can be zero only if each term is zero. If this happens, then all the entries of \( A \) are zero and so \( A \) must be the zero matrix.
(c) This follows by writing out the formulas for \((AB)_{ii}\) and \((BA)_{ii}\); both are equal sums.

\( \square \)

(1.5, 23) (a) Find a \( 2 \times 2 \) idempotent matrix.
(b) Show that

\[
\begin{pmatrix}
-1 & 1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & 1
\end{pmatrix}
\]

is idempotent.
(c) If \( A \) is idempotent, show that \((I - A)\) is also idempotent.
(d) Use the previous two parts to get another idempotent matrix.
(e) Show that \( A \) is idempotent if both \( AB = A \) and \( BA = B \).

Proof. (a) Consider

\[
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix}
\]

(b) This is easily verified by multiplying.
(c) Since \( A^2 = A \), we see that

\[
(I - A)^2 = (I - A)(I - A) = I^2 - 2IA + A^2 = I - 2A + A = I - A.
\]
(d) Subtracting, we obtain
\[
\begin{pmatrix}
2 & -1 & -1 \\
1 & 0 & -1 \\
1 & -1 & 0
\end{pmatrix}
\]

(e) Using the given identities, we obtain
\[A^2 = (AB)A = A(BA) = AB = A.\]

\[\square\]

(2.1, 10) Prove that if more than one solution to a system of linear equations exists, then an infinite number of solutions exists.

Proof. Suppose that \(X_1\) and \(X_2\) are distinct solutions to \(AX = B\). Then for any constant \(c\), we find that
\[A(X_1 + c(X_2 - X_1)) = AX_1 + cAX_2 - AX_1 = B + cAX_2 - AX_1 = B + c(0) = B.\]
That is, \(X_1 + c(X_2 - X_1)\) is also a solution. If \(X_1 + c(X_2 - X_1) = X_1\), then \(c(X_2 - X_1) = 0\) and so \(c = 0\) since \(X_2\) and \(X_1\) are distinct. If \(X_1 + c(X_2 - X_1) = X_2\), then \(c(X_2 - X_1) = X_2 - X_1\) and so \(c = 1\); we can divide since \(X_2 - X_1 \neq 0\). \[\square\]

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