MATH 392: HOMOTOPY TYPE THEORY, PROBLEM SET #3

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1. Problems

(1) In this problem, we will prove the compactness theorem for propositional logic. One version of this states the following: Let $\{F_i\}$ be a set of formulas. Then $\{F_i\}$ is satisfiable if every finite subset of $\{F_i\}$ is satisfiable. Alternatively, we can state this as if $\{F_i\} \models G$ then there exists a finite subset $A \subset \{F_i\}$ such that $A \models G$.
  (a) Prove that these two formulations are equivalent.
  (b) Now prove the first formula of the compactness theorem using Tychonoff's theorem. (Hint: Let $\mathcal{P}$ be the set of all variables that appear in $\{F_i\}$. Consider the space $\{0,1\}^{\mathcal{P}}$.)

(2) Show that the fragment of predicate calculus which does not have equality, quantifiers, or variables has a complete and sound inference procedure. (Hint: reduce to the propositional case by introducing new variables to handle relations.)

(3) This problem develops Heyting algebras.
  (a) Explain how to regard a poset P as a category.
  (b) Let $\mathcal{C}$ be a category arising from a poset which has coproducts, products, a terminal object, and an initial object. Show that $\mathcal{C}$ is cocomplete and complete.
  (c) Assume further that the functor $- \times x$ has a right adjoint, for all $x \in \mathrm{ob}(\mathcal{C})$. Suggestively denote this right adjoint, when evaluated at y, as $x \Rightarrow y$. Show the following arrows exist in $\mathcal{C}$:
    (i) $x \to (x \Rightarrow y)$
    (ii) $x \Rightarrow (y \Rightarrow z) \to (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$
  Further, show that there is a (unique) morphism $x \to y$ if and only if $x \Rightarrow y$ is the initial object.

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