MATH 392: HOMOTOPY TYPE THEORY, PROBLEM SET #4

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1. Problems

(1) Let $\mathcal{C}$ be a category. A monad is a monoid in the category of functors $\mathcal{C} \to \mathcal{C}$, where the product is given by composition. Given a monad M, we denote by $\mathcal{C}[M]$ the category of algebras over M, i.e., objects of $\mathcal{C}$ equipped with a map $MX \to X$ which is associative and unital (with respect to the structure of M).
   (a) Write out explicitly the requirements for a functor to be a monad. Write out what it means to be an algebra over a monad. A comonad is a comonoid in the category of functors; write this explicitly.
   (b) Let (F, G) be adjoint functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$. Show that FG is a comonad and GF is a monad.
   (c) Given an object x in $\mathcal{C}$, Mx is an object of $\mathcal{C}[M]$ (the free M-algebra on x). Explain.
   (d) Given a monad M on $\mathcal{C}$, the Kleisli category $\mathcal{C}_M$ has objects simply the objects of $\mathcal{C}$ and morphisms $X \to Y$ given by maps $X \to MY$. Relate the Kleisli category to the free algebras.
   (e) Define an adjunction $F\colon \mathcal{C} \to \mathcal{C}_M$ and $G\colon \mathcal{C}_M \to \mathcal{C}$ such that the monad GF is precisely M.
   (f) Define the coKleisli category. For a fixed object x in a category $\mathcal{C}$ with finite products, what is the coKleisli category associated to the comonad $(-) \times x$?

(2) Write an expression in the untyped λ-calculus that multiplies Church numerals. Write an expression that computes the predecessor of a Church numeral. Contemplate what would be involved in implementing rational arithmetic.

(3) This problem explores the proof of strong normalization for the simply-typed λ-calculus. (Recall that this theorem says that all expressions in the simply-typed λ-calculus do not give rise to an infinite sequence of $\beta\eta$-reductions.)
   (a) Suppose we wanted to run a simple-minded inductive proof of strong normalization (simply using the structure of the expression). Define G to be set of strongly-normalizing expressions. Explain why even if we assume that $M, N \in G$, we can't necessarily conclude that $MN \in G$.
   (b) Next, for a base type α we define

$$G_\alpha = \{M\colon \alpha \mid M \in G\}$$

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and for arbitrary types β and γ we recursively define

$$G_{\beta\to\gamma} = \{M\colon \beta \longrightarrow \gamma \,|\, MN \in G_\alpha \forall N \in G_\beta\}.$$

Explain why this fixes the problem with application that arose above.

(c) Show by induction that $G_\tau \subseteq G$ for any τ.

(d) We'd now like to show that if $x\colon \beta \models M\colon \gamma$ and $M \in G_\gamma$, then $\lambda x\colon \beta.M \in G_{\beta\to\gamma}$. Why doesn't this just work?

(e) Write a complete proof of strong normalization.

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