Parallax

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1 Parallax

Parallax denotes a simple method of employing trigonometry to determine distances to remote objects using known, local distances and apparent angular deflection. A simple example will help to make the point clear.

Caesar, after one of his campaigns in Gaul, decided to cross into German territory. As the Rhine served as the border between Roman and German territories, he decided to cross the Rhine itself — a particularly wide, deep, and swift-flowing river. In his account of the situation, Caesar states that it was worthy neither of his own dignity or of that of the Roman people that the army should cross on boats. He therefore determined to build a bridge across the river.

To build a bridge, one first needs the distance to be spanned. The Roman army of this era excelled not so much because of its valor in battle, but rather because of its *organization*. This includes in part excellent battle tactics; but in fact the more crucial aspect of military organization turns out to be logistics (realize Napoleon's campaign against Russia failed largely because his supply lines could not feed his army as it roamed deep into foreign territory). *Engineering* forms the crux of military logistics, and in this the Romans excelled.

Thus, Caesar's army must have had available several methods by which to estimate the distance across a river such as the Rhine, but in particular the method of parallax provides a plausible solution for the period. I'm certainly not suggesting that Caesar's engineers did in fact use the method in the Rhine crossing, but they *could* have employed it, as we will see.

Imagine then the following problem as a caricature of the situation Caesar encountered. Caesar charges one of his engineers — call him Marcus — with the task of determining the distance across the Rhine where his army is located. At this point, Marcus has the good fortune to notice one particular

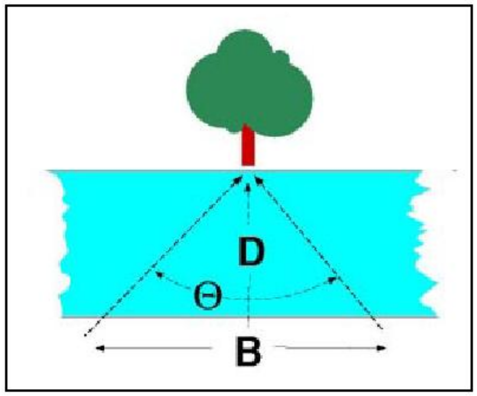

Figure 1: Determining distance via parallax.

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tree on the far shore standing well in front of the distant forest. Marcus determines to use parallax to find the distance to the tree. He proceeds as follows.

Marcus first finds a position on the near shore where his line of sight to the tree is perpendicular to the shoreline. Thus, walking along the shore, he walks perpendicular to the line whose distance he wishes to measure. He then walks 10 meters upstream from the line of sight, and back again; then 10 meters downstream along the shore, and back again. In this way, Marcus sets up his *baseline*, with a total length of 20 meters. He then charges two surveyors with standing at either end of the baseline and sighting the solitary tree against the forest in the background. When the two surveyors return, Marcus compares their measurements.

Example 1.1 (Caesar apud Rhenum)**.** *The measurements of the surveyors show that the tree on the far shore undergoes an angular displacement of* $2.30°$ *when viewed from either end of a baseline measuring* $20\,\mathrm{m}$. *What is the distance to the tree, and hence to the far shore?*

Solution: Figure 1 depicts the situation. Looking at the diagram, we see that the two surveyors' lines of sight, taken with the baseline, form an isosceles triangle. The original line of sight perpendicular to the baseline and directed at the tree divides this isosceles triangle in half, leaving two right triangles of which it forms the base. Figure 2 depicts one of these triangles.

We can now apply basic trigonometry to find the distance d. In particular, we know that the fact that the baseline is 20 m means that the side of the triangle opposite the marked angle has length

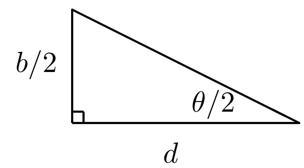

Figure 2: Right triangle.

$$\frac{b}{2} = \frac{20\,\mathrm{m}}{2} = 10\,\mathrm{m}.$$

We then use the relation

$$\tan(\text{angle}) = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{to write} \qquad \tan\left(\frac{\theta}{2}\right) = \frac{b/2}{d}.$$

Multiplying each side by d, we find

$$d = \frac{b/2}{\tan(\theta/2)}.$$

We finally substitute the actual values and calculate

$$d = \frac{20\,\mathrm{m}/2}{\tan(2.30^\circ/2)} = \frac{10\,\mathrm{m}}{\tan(1.15^\circ)} \approx \frac{10\,\mathrm{m}}{0.02} = 500\,\mathrm{m}.$$

Marcus therefore tells Caesar that they require a bridge 500 m long.

End Solution.

As a short postscript, it bears mentioning that not only did Caesar's army manage to construct a 500 m bridge across the Rhine, but they accomplished the task in only 10 days! This serves as a testament to Caesar's logistic aptitude and his engineers' technical skill. Moreover, this evidently came as such a surprise to the Germans that they refused to attack Caesar's army; after some two weeks of uninterrupted exploration of the German countryside, Caesar led his army back to Roman territory and had the bridge dismantled.

2 The Small Angle Approximation

As we can see from the above, parallax has applications which extend far beyond the science of astronomy. However it forms one of the core techniques in the astronomer's arsenal, and it merits close scrutiny. In particular, it finds extensive application in calculating distances to stars at "intermediate" distances.

When measuring the parallax angle of stars, one often encounters very small angles. In general, these angles are far less than 1°, and an arcseconds provide the appropriate unit. With such small angles, one can safely approximate

$$\tan(x) \approx x.$$

We may then write the parallax formula $d = (b/2)/\tan(\theta''/2)$, where θ'' shows that the angle is measured in arcseconds, as

$$d = \frac{b/2}{\tan(\theta''/2)} \approx \frac{b/2}{\theta''/2} = \frac{b}{\theta''}.$$

The above equation, however, cannot stand as it is. The left-hand side d has units of [length], while the right-hand side b/θ'' has units of [length per arcsecond]. The two sides have different units, contradicting the fact that this is an equality. We cannot therefore allow θ to remain expressed in units of [arcsecond], but must convert to [radians], that is θ must be unitless. So we convert units:

$$d = \frac{b}{\theta'' \cdot \left[\frac{1'}{60''}\right] \cdot \left[\frac{1^\circ}{60'}\right] \cdot \left[\frac{2\pi\,\text{radians}}{360^\circ}\right]}.$$

Performing the multiplication in the denominator, we find

$$d \approx \frac{b}{\theta[\text{radians}] \cdot \frac{1}{206265}} = 206265 \cdot \frac{b}{\theta},$$

where now θ is in radians, i.e. unitless. This expression is known as the *small angle approximation* to the parallax formula.