HIGHLY DEGENERATE
HARMONIC MEAN CURVATURE FLOW

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Abstract. We study the evolution of a weakly convex surface Σ in $\mathbb{R}^3$ with flat sides by the Harmonic Mean Curvature flow. We establish the short time existence as well as the optimal regularity of the surface and we show that the boundaries of the flat sides evolve by the Curve Shortening flow. It follows from our results that a weakly convex surface with flat sides of class $C^{k,\gamma}$, for some $k \geq 1$ and $0 < \gamma \leq 1$, remains in the same class under the flow. This distinguishes this flow from other, previously studied degenerate parabolic equations, including the porous medium equation and the Gauss Curvature flow with flat sides, where the regularity of the solution for $t > 0$ does not depend on the regularity of the initial data.

1. Introduction

We consider the motion of a compact, weakly convex two-dimensional surface Σ in space $\mathbb{R}^3$ under the *Harmonic Mean Curvature flow* HMCF

$$\frac{\partial P}{\partial t} = \frac{K}{H}\, N \tag{HMCF}$$

where each point P moves in the inward direction N with velocity equal to the *Harmonic Mean Curvature* of the surface, namely the harmonic mean

$$\frac{K}{H} = \frac{1}{\lambda_1^{-1} + \lambda_2^{-1}}$$

of the two principal curvatures λ_1, λ_2 of the surface.

The existence of solutions to the HMCF with strictly convex smooth initial data was first shown by Andrews in [2] who also showed that under the HMCF strictly convex, smooth surfaces converge to round points in finite time. In [9], Diëter established the short time existence of solutions to the HMCF with weakly convex smooth initial data. More precisely, Diëter showed that if at time $t = 0$ the surface Σ satisfies $K \geq 0$ and $H > 0$, then there exists a unique strictly convex smooth

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solution Σ_t of the HMCF defined on $0 < t < \tau$, for some $\tau > 0$. By the results of Andrews, this solution exists up to the time where its enclosed volume becomes zero. However, in this work, it is not addressed in [9] the highly degenerate case where the initial data is weakly convex and both K and H vanish in a region.

We will consider in this work an initial surface Σ which has flat sides. The parabolic equation describing the motion of the surface becomes degenerate at points where both curvatures K and H become zero. Our main objective is to study the solvability and optimal regularity of the evolving surface Σ_t, for $t > 0$, by viewing the flow as a *free-boundary* problem. It will be shown that a surface Σ of class $C^{k,\gamma}$ with $k \geq 1$ and $0 < \gamma \leq 1$ at $t = 0$ will remain in the same class for $t > 0$. In addtion, we will show that the strictly convex parts of the surface become instantly C^∞ smooth up to the flat sides on $t > 0$ and the boundaries of the flat sides evolve by the Curve Shortening flow.

For simplicity we will assume that the initial surface Σ has only one flat side, namely $\Sigma = \Sigma_1 \cup \Sigma_2$ with Σ_1 flat and Σ_2 strictly convex. Since the equation is invariant under rotation, we may also assume that Σ_1 lies on the $z = 0$ plane and that Σ_2 lies above this plane. Then, the lower part of the surface Σ can be written as the graph of a function

$$z = h(x, y)$$

over a compact domain $\Omega \subset \mathbb{R}^2$ containing the initial flat side Σ_1.

We define $g = h^p$, for some $0 < p < 1$. Let Γ denote the boundary of the flat side Σ_1. Our main assumption on the initial surface Σ is that it satisfies the following non-degeneracy condition, which we call *non-degeneracy condition ($\star$)*:

$$(\star) \qquad |Dg| \geq \lambda \;\text{ and }\; g_{\tau\tau} \geq \lambda, \quad \forall\, P \in \Gamma$$

for some number $\lambda > 0$ and with $q = 1/p$; τ denotes the tangential direction to the level sets of g and $g_{\tau\tau}$ denote the second order derivative in this direction.

Under the above conditions, our main results show that:

a). *The HMCF admits a viscosity solution $\Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$ of class $C^{k,\gamma}$, for some $k \geq 1$ and $0 < \gamma \leq 1$ depending on p, which is smooth up to the interface $\Gamma_t = (\Sigma_1)_t \cap (\Sigma_2)_t$;*

b). *The flat side $(\Sigma_1)_t$ persists for some positive time and, in particular, its boundary Γ_t evolves by the Curve Shortening flow.*

The fact that the solution Σ_t remains in the class $C^{k,\gamma}$, for $t > 0$, distinguishes this flow from other, previously studied, degenerate free-boundary problems (such as the Gauss curvature flow with flat sides, the porous medium equation and the evolution p-laplacian equation) in which the regularity of the solution for $t > 0$ does not depend on the regularity of the initial data.

We define $\mathfrak{S}$ to be the class of weakly convex compact hypersurfaces Σ in $\mathbb{R}^3$ so that $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is a surface contained in the plane $z = 0$ and Σ_2 is a strictly convex and smooth surface contained above the plane $z = 0$. The main result states as follows:

2. Main Theorem

Main Theorem. *Assume that at time $t = 0$, Σ is a weakly convex compact hypersurface in $\mathbb{R}^3$ which belongs to the class $\mathfrak{S}$ so that the function $g = h^p$ defined above is smooth up to the interface Γ and it satisfies the condition $(\star)$. Then, there exists a time $T > 0$ such that the HMCF admits an unique viscosity solution $\Sigma_t \in \mathfrak{S}$ on $[0,T)$. Moreover, the function $g = h^p$, defined as above, is smooth up to the interface $z = 0$ and satisfies condition $(\star)$. In particular, the interface Γ_t between the flat side and the strictly convex side is a smooth curve for all t in $0 < t \leq T$ and it moves by the Curve Shortening flow.*

Sketch of the proof. The HMCF can be seen as a free boundary problem arising from the degeneracy near the flat side of the fully nonlinear parabolic PDE which describes the flow. We can show that this free boundary problem is equivalent to solve an *initial value problem* obtained by introducing an appropriate change of coordinates on $\mathcal{D} \times [0,T]$, with $\mathcal{D} = \{(u,v); u^2 + v^2 \leq 1\}$

$$\begin{cases} Mw = 0 & on\ \mathcal{D} \times [0,T] \\ w = w_0 & at \quad t = 0 \end{cases}$$

$$Mw = w_t - F(t,u,v,w,Dw,D^2 w)$$

is a fully non-linear operator which becomes degenerate at $\partial\mathcal{D}$. To show that this problem admits a solution, one applies the Inverse Function Theorem between appropriately defined Banach spaces.

The linearization of the operator M at a point $\bar{w}$ close to the initial data w_0 can be modeled, after we straighten the boundary, on the degenerate equation

$$f_t = z^2\, a_{11} f_{zz} + 2\, z\, a_{12} f_{zy} + a_{22} f_{yy} + b_1\, z\, f_z + b_2\, f_y \tag{2.1}$$

on $z > 0$, and no extra conditions on f along the boundary $z = 0$. The diffusion in the above equation is hence governed by the Riemannian metric $ds^2 = d\bar{s}^2 + |dt|$ where

$$d\bar{s}^2 = \frac{dz^2}{z^2} + dy^2.$$

Notice that the distance (with respect to the singular metric $\bar{s}$) of an interior point ($z > 0$) from the boundary ($z = 0$) is *infinite*. This distinguishes our problem from other, previously studied, degenerate free-boundary problems such as the degenerate Gauss curvature flow and the porous medium equation.

3. Local Change of Coordinates

We will assume throughout this section that the surface Σ belongs to the class $\mathfrak{S}$. Let Σ_t be a solution to the HMCF on $[0, T]$, for some $T > 0$ in the sense that $\Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$, with $(\Sigma_1)_t$ flat and $(\Sigma_2)_t$ strictly convex. Let $P_0(x_0, y_0, 0)$ be a point on the interface Γ_{t_0}, for $t_0 > 0$ sufficiently small. Then, the strictly convex part of surface $(\Sigma_2)_{t_0}$ can be expressed locally around P_0 as the graph of a function $z = h(x, y, t)$. Let g be defined by $g = h^p$, for $0 < p < 1$. Assume that g is smooth up to the interface and satisfies condition $(\star)$. We can solve locally around the point P_0 the equation $z = h(x, y, t)$ with respect to x. This yields to the map $x = f(z, y, t)$. The hypothesis on g expressed in terms of f give the following *non-degeneracy condition* $(\star\star)$:

$$\begin{pmatrix} -z^{2-p}\, f_{zz} & z^{1-p}\, f_{zy} \\ z^{1-p}\, f_{zy} & -\, f_{yy} \end{pmatrix} \geq \bar{\lambda} \tag{$\star\star$}$$

Since f is the inverse of h and the HMCF is invariant under rotation, the function f satisfies the same equation as h on $z > 0$:

$$f_t = \frac{f_{zz} f_{yy} - f_{zy}^2}{(1 + f_y^2) f_{zz} - 2\, f_z f_y f_{zy} - (1 + f_z^2) f_{yy}} \tag{3.1}$$

By using the Inverse Function Theorem between Banach spaces one can construct a sufficiently smooth solution to this equation. We will define the Banach space

$C_s^{2+\alpha,p}$ in which we solve the Inverse Function Theorem in the next section. According to our notation, the constants α and p indicate "how the surface becomes flat", while s refers to the hyperbolic metric which governs the problem.
Observe that when $f \in C_s^{2+\alpha,p}$ such that it satisfies condition ($\star\star$) the Equation 3.1 becomes degenerate at $z = 0$ implying that:

$$f_t = \frac{f_{yy}}{1 + f_y^2} \quad \text{at the interface } \; z = 0; \tag{3.2}$$

This is equivalent to say that the free boundary Γ_t evolves by the *Curve Shortening flow.*

4. The $C_s^{2+\alpha,p}$ space and Schäuder estimates

Let $\mathcal{A}$ be a compact subset of the half space $\{\,(z,y) \in \mathbb{R}^2 : \; z \geq 0\}$ such that $(0,0) \in \mathcal{A}$. Then, we define:

$$\begin{aligned}
\mathcal{A}^\circ &:= \{\; y \in \mathbb{R} : (0,y) \in \mathcal{A}\,\} \\
\tilde{\mathcal{A}} &:= \{(w,y) \in \mathbb{R}^2 : w = \ln z,\; (z,y) \in \mathcal{A},\; z \neq 0\} \\
Q_T &:= \mathcal{A} \times [0,T],\; T > 0 \\
Q_T^\circ &:= \mathcal{A}^\circ \times [0,T] \\
\tilde{Q}_T &:= \tilde{\mathcal{A}} \times [0,T]
\end{aligned}$$

Let $0 < p < 1$. Given a function f on $\mathcal{A}$ we define:

$$\begin{aligned}
f^\circ(y) &:= f(0,y) \\
\tilde{f}(w,y) &:= e^{-p\,w}\,(\,f(z,y) - f^\circ(y)\,)
\end{aligned}$$

with $w = \ln z$, for $z > 0$.
Analogously, given a function f on Q_T we define:

$$\begin{aligned}
f^\circ(y,t) &:= f(0,y,t) \\
\tilde{f}(w,y,t) &:= e^{-\,p\,w}\,(\,f(z,y,t) - f^\circ(y,t)).
\end{aligned}$$

Given a subspace $\mathcal{A}$ as above, we define the *hyperbolic distance* $\bar{s}(P_1,P_2)$ of two points $P_1 = (z_1,y_1)$ and $P_2 = (z_2,y_2)$ in $\mathcal{A}$ with $z_i > 0$, $i = 1,2$ in the following way:
if ,$0 < z_1,\, z_2 \leq 1$,

$$\bar{s}(P_1,P_2) := \sqrt{|\ln z_1 - \ln z_2|^2 + |y_1 - y_2|^2}$$

otherwise it is equivalent to the standard euclidean metric.

We define the *parabolic hyperbolic distance* between two points $\tilde{P}_1 = (z_1, y_1, t_1)$ and $\tilde{P}_2 = (z_2, y_2, t_2)$ with $z_i > 0$, $i = 1, 2$ to be:

$$s(\tilde{P}_1, \tilde{P}_2) := \bar{s}(P_1, P_2) + \sqrt{|t_1 - t_2|}$$

where $P_1 = (z_1, y_1)$, $P_2 = (z_2, y_2)$.

Let $0 < \alpha \leq 1$. We define the Hölder space $C^{\alpha,p}_{\bar{s}}(\mathcal{A})$ in terms of the above distance. We start defining the Hölder semi-norm:

$$\| f \|_{H^{\alpha}_{\bar{s}}(\mathcal{A})} := \sup_{P_1 \neq P_2 \in \mathcal{A} \cap \{(x,y) \in \mathbb{R}^2 : z > 0\}} \frac{| f(P_1) - f(P_2) |}{s[P_1, P_2]^{\alpha}} .$$

and the norm

$$\| f \|_{C^{\alpha}_{\bar{s}}(\mathcal{A})} := \| f \|_{C^0(\mathcal{A})} + \| f \|_{H^{\alpha}_{\bar{s}}(\mathcal{A})}$$

noindent where $|| f ||_{C^0(\mathcal{A})} := \sup_{P \in \mathcal{A}} | f(P) |$.
We say that a function f belongs to $C^{\alpha,p}_{\bar{s}}(\mathcal{A})$ if $f^{\circ} \in C^{\alpha}(\mathcal{A}^{\circ})$ and $\tilde{f} \in C^{\alpha}(\tilde{\mathcal{A}})$. The norm of f in the space $C^{\alpha,p}_{s}(\mathcal{A})$ is defined as:

$$|| f ||_{C^{\alpha,p}_{\bar{s}}(\mathcal{A})} := || f^{\circ} ||_{C^{\alpha}(\mathcal{A}^{\circ})} + || \tilde{f} ||_{C^{\alpha}(\tilde{\mathcal{A}})}.$$

We introduce the following notation for later purposes: $|| f ||_{C^{0,p}(\mathcal{A})} := || f^{\circ} ||_{C^0(\mathcal{A}^{\circ})} + || \tilde{f} ||_{C^0(\tilde{\mathcal{A}})}$.

Remark 1. *Note that $f(w, y) \in C^{\alpha}(\tilde{\mathcal{A}})$ iff $f(z, y) \in C^{\alpha}_{\bar{s}}(\mathcal{A})$, where $w = \ln z$.*

We say that a continuous function f on $\mathcal{A}$ belongs to $C^{2+p}(\mathcal{A})$ if $f^{\circ} \in C^2(\mathcal{A}^{\circ})$ and f has continuous derivatives

$$f_z,\ f_y,\ f_{zz},\ f_{zy},\ f_{yy}$$

in the interior of $\mathcal{A}$, such that

$$z^{-p}\,(f - f^{\circ}),\ z^{1-p}\,f_z,\ z^{-p}\,(\,f_y - f^{\circ}_y\,),\ z^{2-p}\,f_{zz},\ z^{1-p}\,f_{zy},\ z^{-p}\,(\,f_{yy} - f^{\circ}_{yy}\,)$$

extend continuously up to the boundary. The norm of f in the space $C^{2+p}(\mathcal{A})$ is defined as follows:

$$|| f ||_{C^{2+p}(\mathcal{A})} := || \sum_{m=0}^{2} D^m_y\, f^{\circ} ||_{C^0(\mathcal{A}^{\circ})} + \sum_{m+n=0}^{2} || D^m_z\, D^n_y\, \tilde{f} ||_{C^0(\tilde{\mathcal{A}})}$$

Given $f \in C^{2+p}(\mathcal{A})$, we say that f belongs to $C^{2+\alpha,p}_{\bar{s}}(\mathcal{A})$ if

$$f^{\circ} \in C^{2+\alpha}(\mathcal{A}^{\circ}) \text{ and } z\, f_z, f_y,\, z^2\, f_{zz}, z\, f_{zy}, f_{yy}$$

extend continuously up to the boundary, and the extensions are Hölder continuous on $\mathcal{A}$ of class $C^{\alpha,p}_{\bar{s}}(\mathcal{A})$. The norm of f in the space $C^{2+\alpha,p}_{\bar{s}}(\mathcal{A})$ is defined as:

$$||\, f\,||_{C^{2+\alpha,p}_{\bar{s}}(\mathcal{A})} := ||\, f^{\circ}\,||_{C^{2+\alpha}(\mathcal{A}^{\circ})} + \sum_{m+n=0}^{2} ||\, z^m\, D_z^m\, D_y^n\, f\,||_{C^{\alpha,p}_{\bar{s}}(\mathcal{A})}$$

Remark 2. *By definition:* $\tilde{f}_w = -p\,\tilde{f} + z^{1-p}\, f_z, \quad \tilde{f}_{ww} = -p\,\tilde{f}_z + (1-p)\, z^{2-p}\, f_{zz}$, *hence:*

$$\sum_{m+n=0}^{2} ||\, z^m\, D_z^m\, D_y^n\, f\,||_{C^{\alpha,p}_{\bar{s}}(\mathcal{A})} \simeq ||\,\tilde{f}\,||_{C^{2+\alpha}(\tilde{\mathcal{A}})}$$

Remark 3. *The function* $f \in C^{2+\alpha,p}_{\bar{s}}(\mathcal{A})$ *iff* $f^{\circ} \in C^{2+\alpha}(\mathcal{A}^{\circ})$ *and* $\tilde{f} \in C^{2+\alpha}(\tilde{\mathcal{A}})$, *therefore, the following norms are equivalent:*

$$||\, f\,||_{C^{2+\alpha,p}_{\bar{s}}(\mathcal{A})} \simeq ||\, f^{\circ}\,||_{C^{2+\alpha}(\mathcal{A}^{\circ})} + ||\,\tilde{f}\,||_{C^{2+\alpha}(\tilde{\mathcal{A}})}$$

Let $T > 0$. The definitions above can be naturally extended on the space-time domain Q_T by using the parabolic distance $ds^2 = d\bar{s}^2 + |dt|$. We define the space $C^{\alpha}_s(Q_T)$ to be the standard Hölder space with respect to the metric ds^2. *We say that a continuous function f on Q_T belongs to $C^{2+p}(Q_T)$ if f has continuous derivatives*

$$f_t, f_z, f_y, f_{zz}, f_{zy}, f_{yy}$$

in the interior of Q_T and f° has continuous derivatives that extend continuously up to the boundary and

$$z^{-p}\,(f - f^{\circ}),\ z^{-p}\,(f_t - f^{\circ}_t),\ z^{1-p} f_z, z^{-p}\, f_y,\ z^{2-p} f_{zz},\ z^{1-p} f_{zy},\ z^{-p}\,(f_{yy} - f^{\circ}_{yy})$$

extend continuously up to the boundary. The norm of f in the space $C^{2+p}(Q_T)$ is defined as follows:

$$||\, f\,||_{C^{2+p}} := ||\, f^{\circ}\,||_{C^2} + \sum_{l+m+2j=0}^{2} ||\, D_z^l\, D_y^m\, D_t^j\,\tilde{f}||_{C^{\circ}}$$

The function f belongs to $C^{2+\alpha,p}_s(Q_T)$ if $f \in C^{2+p}(Q_T)$ and

$$f, f_t, zf_z, f_y \qquad \textit{and} \qquad z^2 f_{zz}, zf_{zy}, f_{yy} \ \textit{belong to}\ C^{\alpha,p}_s(Q_T).$$

Let k be a positive integer. We can extend these definitions to spaces of higher order derivatives. We denote by $C^{k,p}(Q_T)$ the space of all functions f whose k-th order derivatives $D_z^i\, D_y^j\, D_t^l\, f$, $i+j+2l = k$ in the interior of Q_T and $z^i\, D_z^i\, D_y^j D_t^l\,(f-$

f°), $i+j+2l=k$ exist and belong to the space $C^0(Q_T)$. We define $C^{\infty,p}(Q_T) = \cap_k C^{k,p}(Q_T)$.

We denote by $C_s^{k+\alpha,p}(Q_T)$ the space of all functions $f \in C^{k,p}(Q_T)$ such that $z^i\, D_z^i\, D_y^j\, D_t^l\, f$, $i+j+2l=k$ belong to the space $C_s^{\alpha,p}(Q_T)$. This space $C_s^{k+\alpha,p}(Q_T)$ is equipped with the norm:

$$|| \, f \, ||_{C_s^{k+\alpha,p}(Q_T)} := \sum_{i+j+2l \le k} || \, z^i \, D_z^i D_y^j D_t^l \, f \, ||_{C_s^{\alpha,p}(Q_T)}.$$

Remark 4. *A function $f \in C_s^{k+\alpha,p}(Q_T)$ iff $f^\circ \in C^{k+\alpha}(Q_T^\circ)$ and $\tilde{f} \in C^{k+\alpha}(\tilde{Q}_T)$. Moreover,*

$$|| \, f \, ||_{C_s^{k+\alpha,p}(Q_T)} \simeq || \, f^\circ \, ||_{C^{k+\alpha,[k/2]+\alpha/2}(Q_T^\circ)} + || \, \tilde{f} \, ||_{C^{k+\alpha,[k/2]+\alpha/2}(\tilde{Q}_T)}.$$

In the next paragraph we denote by $\mathcal{S}_0$ the half space $x \ge 0$ in $\mathbb{R}^2$, by $\mathcal{S}$ the space $\mathcal{S} = \mathcal{S}_0 \times [0,\infty)$, and by $\mathcal{S}_T$ the space $\mathcal{S} \times [0,T]$, for $T > 0$. The operator $L_k : C_s^{k+2+\alpha,p}(Q_T) \to C_s^{k+\alpha,p}(Q_T)$ is defined as:

$$L\,[\,f\,] := f_t - (\, z^2 a_{11} f_{zz} + 2\, z\, a_{12} f_{zy} + a_{22} f_{yy} + b_1 z\, f_z + b_2 f_y + c\, f\,) \tag{4.1}$$

where the coefficients $\{a_{ij}\}$ are uniformly elliptic and $\{a_{ij},\, b_i,\, c\} \subseteq C_s^{k+2+\alpha}(Q_T)$, $\{a_{22},\, b_2,\, c\} \subseteq C_s^{k+2+\alpha,p}(Q_T)$.

Theorem 1. *(Existence and Uniqueness) Let L be as above. Assume that $\phi \in C_s^{k+\alpha,p}(\mathcal{S})$ and $f^\circ \in C_s^{k+2+\alpha,p}(\mathcal{S}_0)$, and both ϕ, f_0 are compactly supported in $\mathcal{S}$ and $\mathcal{S}_0$ respectively. Then, for any $T>0$, the initial value problem*

$$\begin{cases} L\,[f] &= \phi \quad \text{in } S_T \\ f(\cdot,0) &= f_0 \quad \text{on } S_0 \end{cases} \tag{4.2}$$

admits a unique solution $f \in C_s^{k+2+\alpha,p}(\mathcal{S}_T)$. Moreover

$$||f||_{C_s^{k+2+\alpha,p}(\mathcal{S}_T)} \le C(T)\left(||f_0||_{\mathcal{C}_s^{k+2+\alpha,p}(\mathcal{S}_0)} + ||\phi||_{\mathcal{C}_s^{k+\alpha,p}(\mathcal{S})}\right) \tag{4.3}$$

for some constant $C(T)$, depending only α, k and T.

Proof. It is easy to observe that the interesting case holds when the Lebesgue measure of the supports of f_0° and ϕ° satisfies $|(Supp\, f_0)^\circ| \ge \eta$, $|(Supp\, \phi)^\circ| \ge \eta$, for some $\eta > 0$.

To solve the above Cauchy problem is equivalent to solve the following Cauchy problems (4.4) and (4.5):

(4.4) is obtained by evaluating (4.2) at $z = 0$ and (4.5) is obtained by solving the corresponding problem for $\tilde{f}$.

$$\begin{cases} L_0\,[f^\circ] = \phi^\circ & in\ \mathbb{R} \times [0, T] \\ f^\circ(\cdot, 0) = (f_0)^\circ & on\ \mathbb{R} \end{cases} \tag{4.4}$$

$$\begin{cases} \tilde{L}\,[\tilde{f}] = \tilde{\phi} & in\ S_T \\ \tilde{f}(\cdot, 0) = \tilde{f}_0 & on\ S_0 \end{cases} \tag{4.5}$$

where the operators L_0 and $\tilde{L}$ are defined respectively as follows:

$$L_0(f^\circ) = a^\circ_{22}\, f^\circ_{yy} + b^\circ_2\, f^\circ_y + c^\circ f^\circ;$$

$$\tilde{L}\,[\tilde{f}] = \tilde{f}_t - (\hat{a}_{11}\tilde{f}_{ww} + 2\,\hat{a}_{12}\tilde{f}_{wy} + \hat{a}_{22}\tilde{f}_{yy} + \hat{b}_1\,\tilde{f}_w + \hat{b}_2\,\tilde{f}_y + \hat{c}\,\tilde{f} + \hat{G}) \tag{4.6}$$

with

$$\begin{aligned} \hat{a}_{ij}(w, y, t) &:= a_{ij}(x, y, t) \\ \hat{b}_1(w, y, t) &:= (2p - 1)\, a_{11}(w, y, t) + b_1(x, y, t) \\ \hat{b}_2(w, y, t) &:= b_2(x, y, t) \end{aligned}$$

$$\begin{aligned} \hat{c}(w, y, t) &:= e^{-p\,z}[\,p^2\,\hat{a}_{11}(x, y, t) - 2\,p\,\hat{a}_{12}(w, y, t) + p\,b_1(x, y, t)\,] \\ \hat{G}(w, y, t) &:= \tilde{b_2}(w, y, t)\, g^\circ_y(y, t) + \hat{a}_{22}(w, y, t) g^\circ_{yy}(y, t). \end{aligned}$$

By the assumptions on the operator L it is clear that the coefficients of the two operators L_0 and $\tilde{L}$ satisfy classical conditions. Hence, the way we proceed is the following: we first find the solution f° to (4.4), then we solve (4.5). By classical theory both problems have a unique solution. The function f defined by $f(w, y, t) := f^\circ(y, t) + z^p\,\tilde{f}(w, y, t)$ is a solution to (4.2).

Moreover, since the following inequalities hold:

$$||f^\circ||_{C^{k+2+\alpha}(\mathbb{R}^+ \times [0,T])} \leq C(T)\left(||f^\circ_0||_{C^{k+2+\alpha}(\mathbb{R}^+)} + ||g^\circ||_{\mathcal{C}^{k+\alpha}(\mathbb{R}^+)}\right)$$

$$||\tilde{f}||_{C^{k+2+\alpha}(\tilde{\mathcal{S}}_T)} \leq C(T)\left(||\tilde{f}_0||_{C^{k+2+\alpha}(\tilde{\mathcal{S}}_0)} + ||\tilde{\phi}||_{C^{k+\alpha}(\tilde{\mathcal{S}})}\right)$$

where $C^{k+\alpha}$ and $C^{k+2+\alpha}$ denote classical parabolic Hölder spaces, it follows that the solution to (4.2) is unique and it satisfies the inequality (4.3). □

Next, we define the boxes in which we prove the Schaüder estimates. Let $0 < r < 1$. We denote by $\mathcal{B}_r(P)$ the box

$$\mathcal{B}_r(P) = \left\{ \begin{pmatrix} z \\ y \\ t \end{pmatrix} : \begin{matrix} z \geq 0, |x - x_0| \leq e^r \\ |y - y_0| \leq r \\ t_0 - r^2 \leq t \leq t_0 \end{matrix} \right\}$$

around the point $P = \begin{pmatrix} z_0 \\ y_0 \\ t_0 \end{pmatrix}$ and we let $\mathcal{B}_r$ be the box around the point $P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Remark 5. *The choice of the box $\mathcal{B}$ is made so that it has the right rescaling. The operators L_0 and $\tilde{L}$ are well understood on the corresponding boxes $\mathcal{B}_r^\circ$ and $\tilde{\mathcal{B}}_r$.*

Let $k \geq 0$.

Theorem 2. *(Schäuder Estimate) Assume that all the coefficients of the operator*

$$L\,f = f_t - (z^2\,a_{11} f_{zz} + 2\,z\,a_{12} f_{zy} + a_{22} f_{yy} + z\,b_1 f_x + b_2 f_y + \,cf)$$

belong to the space $C_s^{\alpha+k}(\mathcal{B}_1)$ and that the coefficients a_{22}, b_2 and c belong to $C_s^{k+\alpha,p}$ for some numbers α, p in $0 < p < 1$, $0 < \alpha \leq 1$ and satisfy

$$a_{ij}\xi^i\xi^j \geq \lambda|\xi|^2, \ \ \forall \xi \in \mathcal{R}^2 \setminus \{0\}$$

$$||a_{ij}||_{C_s^{k+\alpha}(Q_T)}, \, ||b_i||_{C_s^{k+\alpha}(Q_T)}, \, ||a_{22}||_{C_s^{k+\alpha,p}(Q_T)}, ||b_2||_{C_s^{k+\alpha,p}(Q_T)}, ||c||_{C_s^{k+\alpha,p}(Q_T)} \leq \frac{1}{\lambda}.$$

Then, there exists a constant C depending only on α, λ and p such that

$$\|f\|_{C_s^{2+\alpha,p}(\mathcal{B}_{1/2})} \leq C\left(\|f\|_{C^{0,p}(\mathcal{B}_1)} + \|L[f]\|_{C_s^{\alpha,p}(\mathcal{B}_1)}\right)$$

for all functions $f \in C_s^{2+\alpha,p}(\mathcal{B}_1)$.

Proof. The proof follows by the same argument as in Theorem 1 and classical Schäuder estimates for strictly parabolic operator. □

5. The Degenerate Equation on the disc

We will extend the existence and uniqueness the Theorem 1 to the following class of linear degenerate equations:

$$Lw := w_t - \left(a^{ij} w_{ij} + b^i\, w_i + c\, w \right)$$

on the cylinder $\mathcal{D} \times [0,T)$, $T > 0$, where $\mathcal{D}$ denotes the unit disk in $\mathbb{R}^2$. The sub-indices $i, j \in \{x, y\}$ denote differentiation with respect to the space variables x, y and the summation convention is used. The matrix $\{a^{ij}\}$ is assumed to be symmetric. Certain assumptions on the coefficients will be made so that this class of equations includes, under appropriate change of coordinates, the operator (4.1).

We define the distance function s in $\mathcal{D}$ as follows: in the interior of $\mathcal{D}$, $\bar{s}$ it is equivalent to the standard euclidean distance, while around any boundary point $P \in \partial\mathcal{D}$, $\bar{s}$ is defined as the pull back of the distance function induced by the metric

$$d\bar{s}^2 = \frac{dz^2}{z^2} + dy^2$$

on the half space $\mathcal{S}_0 = \{(z, y) : z \geq 0\}$, via a map $\varphi : \mathcal{S}_0 \cap \mathcal{D} \to \mathcal{D}$ that flattens the boundary of the disk $\mathcal{D}$ near P.

The parabolic distance is defined by

$$s\left[\begin{pmatrix} P_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} P_2 \\ t_2 \end{pmatrix}\right] = \bar{s}(P_1, P_2) + \sqrt{|t_1 - t_2|}.$$

We can now define the spaces $C_s^{k+\alpha,p}(\mathcal{D})$ and $C_s^{k+2+\alpha,p}(\mathcal{D})$: for a fixed small number δ in $0 < \delta < 1$, we write

$$\mathcal{D} = \mathcal{D}_{1-\delta/2} \cup \big(\bigcup_l (\mathcal{D}_\delta(P_l) \cap \mathcal{D})\big)$$

for finite many points $P_l \in \partial\mathcal{D}$, $l \in I$, with $\mathcal{D}_{1-\delta/2}$ denoting the disk centered at the origin of radius $1 - \delta/2$ and $\mathcal{D}_\delta(P_l)$ denoting the disk of radius δ centered at P_l.

We denote by $\mathcal{D}_+$ the half disk

$$\mathcal{D}_+ = \{\, (x, y) \in \mathcal{D} : \; x \geq 0 \,\}.$$

We can choose charts $\Upsilon_l : \mathcal{D}_+ \to \mathcal{D}_\delta(x_l) \cap \mathcal{D}$ which flatten the boundary of $\mathcal{D}$ and such that $\Upsilon_l(0) = P_l$. Let ψ, ψ_l be a partition of unity subordinated to the cover

$$\{\, \mathcal{D}_{1-\delta/2}, \, (\mathcal{D}_\delta(P_l) \cap \mathcal{D}) \,\}$$

of $\mathcal{D}$.

For any integer $k \geq 0$, we define $C_s^{k+\alpha,p}(\mathcal{D})$ to be the space of all functions w on $\mathcal{D}$ such that $w \in C^{k+\alpha}(\mathcal{D}_{1-\delta/2})$ and $w \circ \Upsilon_l \in C_s^{k+\alpha,p}(\mathcal{D}_+)$ for all $l \in I$.

Also, we define $C_s^{k+2+\alpha,p}(\mathcal{D})$ to be the space of all functions w on $\mathcal{D}$ such that $w \in C^{k+2+\alpha}(\mathcal{D}_{1-\delta/2})$ and $w \circ \psi_l \in C_s^{k+2+\alpha,p}(\mathcal{D}_+)$ for all $l \in I$. Here $C^{k+\alpha}$ and $C^{k+2+\alpha}$ denote the regular Hölder Spaces, while $C_s^{k+\alpha,p}(\mathcal{D}_+)$ and $C_s^{k+2+\alpha,p}(\mathcal{D}_+)$

denote the Hölder Spaces defined in section 4. One can show that both spaces $C_s^{k+\alpha,p}(\mathcal{D})$ and $C_s^{k+2+\alpha,p}(\mathcal{D})$ are Banach Spaces under the norms

$$||w||_{C_s^{k+\alpha,p}(\mathcal{D})} = ||\psi\, w||_{C^{k+\alpha}(\mathcal{D}_{1-\delta/2})} + \sum_l ||\psi_l\, (w \circ \Upsilon_l)||_{C_s^{k+\alpha,p}(\mathcal{D}_+)}$$

and

$$||w||_{C_s^{k+2+\alpha,p}(\mathcal{D})} = ||\psi\, w||_{C^{k+2+\alpha}(\mathcal{D}_{1-\delta/2})} + \sum_l ||\psi_l\, (w \circ \Upsilon_l)||_{C_s^{k+2+\alpha,p}(\mathcal{D})}.$$

The above definitions can be extended in a straight forward manner to the parabolic spaces $C_s^{\alpha,p}(Q)$ and $C_s^{2+\alpha,p}(Q)$ where Q is the cylinder $Q = \mathcal{D} \times [0,T]$, for some $T > 0$. Before we state the main result in this section, we will give the assumptions on the coefficients of the equation

$$w_t = a^{ij} w_{ij} + b^i\, w_i + c\, w$$

on the cylinder $Q = \mathcal{D} \times [0,T)$, $i,j = 1,2,$.

We first assume that for any δ in $0 < \delta < 1$, the coefficients $\{a^{ij}\}$, b^i and c belong to the Hölder class $C^\alpha(\mathcal{D}_{1-\delta/2} \times [0,T])$, which means that the coefficients are of the class C^α in the interior of $\mathcal{D}$. For a number δ in $0 < \delta < 1$, let $\Upsilon_l : \mathcal{D}_+ \to \mathcal{D}_\delta(P_l) \cap \mathcal{D}$ be the collection of charts which flatten the boundary of $\mathcal{D}$, considered above. We assume that there exists a number δ so that for every $l \in I$, the coordinate change introduced by each of the Υ_l transforms the operator

$$L[w] = w_t - (\, a^{ij}\, w_{ij} + b^i\, w_i + c\, w\,) \tag{5.1}$$

on $\mathcal{D}_\delta(P_l) \cap \mathcal{D}$, into an operator $\widetilde{L}_l$ on $\mathcal{D}_+$ of the form

$$\widetilde{L}_l\,[\tilde{w}] = \tilde{w}_t - (\, x^2\, \tilde{a}_{11}\, \tilde{w}_{xx} + 2\, x\, \tilde{a}_{12}\, \tilde{w}_{xy} + \tilde{a}_{22}\, \tilde{w}_{yy} + x\, \tilde{b}_1\, \tilde{w}_x + \tilde{b}_2\, \tilde{w}_y \, + \tilde{c}\, \tilde{w})$$

with the coefficients $\tilde{a}_{ij}$, $\tilde{b}_i$ and $\tilde{c}$ belonging to the class $C_s^{k+\alpha}(\mathcal{D}_+)$, with a_{22}, b_2 and $c \in C_s^{k+\alpha,p}$ such that:

$$\tilde{a}_{ij}\xi^i\xi^j \geq \lambda\, |\xi|^2, \qquad \forall \xi \in \mathcal{R}^2 \setminus \{0\}$$

for some number $\lambda > 0$.

We need the next Lemma to prove the invertibility of the operator L:

Lemma 3. *(Hölder Interpolation). For every $\epsilon > 0$ there exists a constant $C(\epsilon)$ depending on ϵ, p, k and α such that for any $g \in C_s^{k+2+\alpha,p}(Q_\delta)$, the following* *inequality holds:*

$$||\vartheta\, D\, g\,||_{C_s^{k+\alpha,p}(Q_\delta)} \leq \epsilon\, ||\, g\,||_{C_s^{k+2+\alpha,p}(Q_\delta)} + C(\epsilon)\, ||\, g\,||_{C^{k,p}(Q_\delta)}. \tag{5.2}$$

where ϑ behaves like distance to the boundary.

Proof. The above interpolation follows by standard arguments. □

The following existence result readily follows from Theorem 1 and the above discussion.

Theorem 4. *Assume that the operator L satisfies all the above conditions on the cylinder $Q = \mathcal{D} \times [0, T]$. Then, given any function $w^0 \in C_s^{k+2+\alpha,p}(\mathcal{D})$ and any function $g \in C_s^{k+\alpha,p}(Q)$ there exists a unique solution $w \in C_s^{k+2+\alpha,p}(Q_T)$ of the initial value problem*

$$\begin{cases} Lw = g & \text{in } Q \\ w(\cdot, 0) = w^0 & \text{on } \mathcal{D} \end{cases}$$

satisfying

$$||w||_{C_s^{k+2+\alpha,p}(Q)} \leq C(T) \left(||w^0||_{C_s^{k+2+\alpha,p}(\mathcal{D})} + ||g||_{C_s^{k+\alpha,p}(Q)} \right) \tag{5.3}$$

The constant $C(T)$ depends only on the numbers α, k, λ and T.

Proof. We can assume, without loss of generality, that $w^0 \equiv 0$ and that g is a function in $C_s^{k+\alpha,p}(Q_T)$, which vanishes at $t = 0$.

For $\delta > 0$, set $Q_\delta = \mathcal{D} \times [0, \delta]$ and denote by $C_{s,0}^{k+2+\alpha,p}(Q_\delta)$ and $C_{s,0}^{k+\alpha,p}(Q_\delta)$ the subspaces of $C_s^{k+2+\alpha}(Q_\delta)$ and $C_s^{k+\alpha}(Q_\delta)$ respectively, consisting out of all functions which vanish identically at $t = 0$. Also, we denote by I the identity operator on $C_{s,0}^{k+\alpha,p}(Q_\delta)$. We will show that, if δ is sufficiently small, there exists an operator $M : C_{s,0}^{k+\alpha,p}(Q_\delta) \to C_{s,0}^{k+2+\alpha,p}(Q_\delta)$ such that

$$||\, L\, M - I\, || \leq \frac{1}{2}.$$

This implies that the operator $L\, M : C_{s,0}^{k+\alpha,p}(Q_\delta) \to C_{s,0}^{k+\alpha,p}(Q_\delta)$ is invertible and therefore $L : C_{s,0}^{k+2+\alpha,p}(Q_\delta) \to C_{s,0}^{k+\alpha,p}(Q_\delta)$ is onto, as desired.

We begin by expressing the compact domain $\mathcal{D}$ as the finite union

$$\mathcal{D} = \mathcal{D}_0 \cup \bigcup_{l \geq 1} \mathcal{D}_l$$

of compact domains in such a way that

$$\operatorname{dist}(\mathcal{D}_0, \partial\mathcal{D}) \geq \frac{\rho}{2} > 0$$

and for all $l \geq 1$

$$\mathcal{D}_l = B_\rho(x_l) \cap \mathcal{D}$$

with $B_\rho(x_l)$ denoting the ball centered at $x_l \in \partial\mathcal{D}$ of radius $\rho > 0$. The number $\rho > 0$ will be determined later.

The operator L is non-degenerate when restricted on the interior domain $\mathcal{D}_0$. Therefore, the classical Schäuder theory for linear parabolic equations implies that L is invertible when restricted on functions which vanish outside $\mathcal{D}_0$.

We denote by $M_0 : C^{k+\alpha,p}_{s,0}(\mathcal{D}_0 \times [0,\delta]) \to C^{k+2+\alpha,p}_{s,0}(\mathcal{D}_0 \times [0,\delta])$ the inverse of the operator L restricted on $\mathcal{D}_0$. Next, we consider the domains $\mathcal{D}_l$, $l \geq 1$, close to the boundary of $\mathcal{D}$, which can be chosen in such a way that the sets $B_{\rho/4}(x_l) \cap \mathcal{D}$ are disjoint. Denoting by $\overline{B}$ the half unit ball

$$\overline{B} = \{(x,y) \in B_1(0)\, ;\ \ x \geq 0\,\}$$

and by $\overline{Q}_\delta$ the cylinder

$$\overline{Q}_\delta = \overline{B} \times [0,\delta]$$

we select smooth charts $\Upsilon_l : \overline{B} \to \mathcal{D}_l$, which flatten the boundary of $\mathcal{D}$, i.e., they map $\overline{B} \cap \{x = 0\}$ onto $\mathcal{D}_l \cap \partial\mathcal{D}$ and have $\Upsilon_l(0) = x_l$. This is possible if the number ρ is chosen sufficiently small. Under the change of coordinates induced by the charts Υ_l, the operator L, restricted on each $\mathcal{D}_l \times [0,\delta]$, is transformed to an operator $\bar{L}_l$ of the form

$$\bar{L}_l[\bar{w}] = \bar{w}_t - (\, x^2\, \bar{a}_l^{11} \bar{w}_{11} + 2\, x\, \bar{a}_l^{12} \bar{w}_{12} + \bar{a}_l^{22} \bar{w}_{22} + \, x \bar{b}_l^1\, \bar{w}_1 + \bar{b}_l^2\, \bar{w}_2 + \bar{c}_l\, \bar{w})$$

defined on $\overline{B} \times [0,\delta]$. Moreover, the charts Υ_l can be chosen appropriately so that the coefficients of $\bar{L}_l$ satisfy

$$\bar{a}_l^{ij} \xi_i \xi_j \geq \lambda |\xi|^2 > 0 \qquad \forall \xi \in \mathbb{R}^2 \setminus \{0\}$$

and

$$||\bar{a}_l^{ij}||_{C_s^{k+\alpha,p}(\bar{Q}_\delta)} \quad ||\bar{b}_l^i||_{C_s^{k+\alpha,p}(\bar{Q}_\delta)} \quad ||\bar{c}_l||_{C_s^{k+\alpha,p}(\bar{Q}_\delta)} \leq 1/\,\bar{\lambda}$$

for some positive constant $\bar{\lambda}$.

Each of the operators $\bar{L}_l$ has the form of the model operators previously studied. Denote by S_0 the half space $x \geq 0$ in $\mathbb{R}^2$ and by S_δ the space $\mathcal{S}_0 \times [0,\delta]$. Also, consider the subspace $\overline{C}^{k+\alpha,p}_{s,0}(\mathcal{S}_\delta)$ of $C^{k+\alpha,p}_{s,0}(\mathcal{S}_\delta)$, consisting out of functions which are compactly supported on $\mathcal{S}_\delta$. Then, Theorem 1 implies that for every $l = 1, 2, ...$ there is an operator $\overline{M}_l : \overline{C}^{k+\alpha,p}_{s,0}(S_\delta) \to C^{k+2+\alpha,p}_{s,0}(S_\delta)$ such that

$$\bar{L}_l\, \overline{M}_l = I$$

with I denoting the identity operator on $\overline{C}^{k+\alpha,p}_{s,0}(\mathcal{S}_\delta)$. Let M_l be the pull back of the operator $\overline{M}_l$ via the chart Υ_l. Next, choose a nonnegative partition of unity ϕ_l; $l = 0, 1, ...$ subordinated to the cover $\mathcal{D}_l$; $l = 0, 1, ...$ of $\mathcal{D}$ and also choose, for each $l \geq 0$, nonnegative, smooth bump functions ψ_l, $0 \leq \psi_l \leq 1$, supported in $\mathcal{D}_l$ with $\psi_l \equiv 1$ on the support of ϕ_l. Then $\sum_{l\geq 0} \phi_l = 1$ and $\psi_l\, \phi_l = \phi_l$ for all l.

We aim to show that the operator $M : C^{k+\alpha,p}_{s,0}(Q_\delta) \to C^{k+2+\alpha,p}_{s,0}(Q_\delta)$ defined as

$$M g \;=\; \sum \psi_l M_l\, \phi_l g$$

satisfies

$$||\, LM\, g \;-\; g\,||_{C^{k+\alpha,p}_s(Q_\delta)} < \frac{1}{2}\; ||g||_{C^{k+\alpha,p}_s(Q_\delta)} \qquad \forall g \in C^{k,\alpha,p}_{s,0}(Q_\delta)$$

if the cover $\{\mathcal{D}_l\}$ and δ are chosen appropriately. Indeed, we can write

$$L\, M g - g = \sum_l L\, \psi_l M_l\, \phi_l g - \sum_l \phi_l g = \sum_l\; \psi_l\, (L M_l - I)\, \phi_l g + \sum_l\, [\, L, \psi_l\,]\, M_l\, \phi_l g$$

with $[\, L, \psi_l\,]$ denoting the commutator of L and ψ_l. The commutator $[\, L, \psi_l\,]$ is only of first order and it can be estimated as

$$||[\, L, \psi_l\,]\, M_l \phi_l g||_{C^{k+\alpha,p}_s(Q_\delta)} \leq C\, \Big(\, ||\vartheta\, D(M_l\phi_l g)||_{C^{k+\alpha,p}_s(Q_\delta)} + \, ||M_l\phi_l g||_{C^{k+\alpha,p}_s(Q_\delta)} \Big)\, .$$

Let $\epsilon > 0$. It follows via the Hölder spaces interpolation from Lemma 3 that

$$||\vartheta\, D(M_l\phi_l g)||_{C^{k+\alpha,p}_s(Q_\delta)} \leq \epsilon\, ||M_l\phi_l g||_{C^{k+2+\alpha,p}_s(Q_\delta)} + C(\epsilon)\, ||M_l\phi_l g||_{C^{k,p}(Q_\delta)}.$$

However, for each k we have

$$||M_l\phi_l\; g||_{C^{k+2+\alpha,p}_s(Q_\delta)} \leq C\, ||g||_{C^{k,\alpha,p}_s(Q_\delta)}$$

and therefore, since $M_l\phi_l g \equiv 0$ at $t = 0$,

$$||M_l\phi_l g||_{C^{k,p}_s(Q_\delta)} \leq C\, \delta\, ||g||_{C^{k+\alpha,p}_s(Q_\delta)}.$$

It follows that if we choose δ sufficiently small:

$$\sum_l ||\,[\,L, \psi_l\,]\,M_l \phi_l g\,||_{C_s^{k+\alpha,p}(Q_\delta)} \leq \frac{1}{4}\,||g||_{C_s^{k+\alpha,p}(Q_\delta)}.$$

On the other hand we have $(\,LM_0 - I\,)\varphi_0 g = 0$, while for $l \geq 1$, we can make the norm of each of the operators $LM_l - I$ arbitrarily close to zero by choosing the diameters of the domains $\mathcal{D}_l$ sufficiently small:

$$||\,\sum_l \psi_l\,(L\,M_l - I)\,\phi_l g\,||_{C_s^{k+\alpha,p}(Q_\delta)} < \frac{1}{4}\,||g||_{C_s^{k+\alpha,p}(Q_\delta)}$$

for all $g \in C_s^{k+\alpha,p}(Q_\delta)$, if ρ and δ are both sufficiently small. The above inequalities give:

$$||\,L\,Mg - g\,||_{C_s^{k+\alpha,p}(Q_\delta)} \leq \frac{1}{2}\,||g||_{C_s^{k+\alpha,p}(Q_\delta)}$$

for all $g \in C_{s,0}^{k+\alpha,p}(Q_\delta)$. We conclude that for every $g \in C_{s,0}^{k+\alpha,p}(Q_\delta)$ there exists a function $w \in C_{s,0}^{k+2+\alpha}(Q_\delta)$ such that $Lw = g$. In addition

$$||w||_{C_s^{k+2+\alpha,p}(Q_\delta)}\;\leq\;C\,||g||_{C_s^{k+\alpha,p}(Q_\delta)} \tag{5.4}$$

with C depending only on $\mathcal{D}$ and the constants α, k, λ and T.

The last inequality implies we can extend the solution on a bigger interval. Hence, one can show that

$$||w||_{C_s^{k+2+\alpha,p}(Q)} \leq C(T)\left(||w^0||_{C_s^{k+2+\alpha,p}(\mathcal{D})} + ||g||_{C_s^{k+\alpha,p}(Q)}\right)$$

where the constant $C(T)$ depends only on the numbers α, k, λ and T.
This last inequality implies uniqueness. □

Finally, the following existence result follows from Theorem 4 and the Inverse Function Theorem between Banach spaces along the line of the proof of Theorem 7.3 in [8]:

Theorem 5. *Let w^0 be a function in $C_s^{k+2+\alpha,p}(\mathcal{D})$. Assume that the linearization $DM(\bar{w})$ of the fully-nonlinear operator*

$$Mw = w_t - F(t,u,v,w,Dw,D^2w)$$

defined on the cylinder $Q = \mathcal{D}\times[0,T]$, satisfies the hypotheses of Theorem 4 at all points $\bar{w} \in C_s^{k+2+\alpha,p}(Q)$, with $||\bar{w} - w^0||_{C_s^{k+2+\alpha,p}(Q)} \leq \mu$, for some $\mu > 0$. Then, there exists a number τ_0 in $0 < \tau_0 \leq T$ depending on the constants α, p, k, λ and

μ, for which the initial value problem

$$\begin{cases} w_t = F(t,u,v,w,Dw,D^2w) & \text{in } \mathcal{D}\times[0,\tau_0] \\ w(\cdot,0)=w^0 & \text{on } \mathcal{D} \end{cases}$$

admits a solution w in the space $C_s^{k+2+\alpha,p}(\mathcal{D}\times[0,\tau_0])$. Moreover,

$$||w||_{C_s^{k+2+\alpha,p}(\mathcal{D}\times[0,\tau_0])} \le C\,||w^0||_{C_s^{k+2+\alpha,p}(\mathcal{D})}$$

for some positive constant C which depends only on α, p, k, λ and μ.

6. Global change of coordinates and existence in $C_s^{k+2+\alpha,p}$

In this section we introduce a global change of coordinates which transforms the HMCF into a fully-nonlinear degenerate parabolic PDE on $\mathcal{D}$.

Let S be a smooth surface close to Σ_2. Let $S:\mathcal{D}\to\mathbb{R}^3$, be a parameterization of S on the unit disk $\mathcal{D}$, $S(u,v)=(x,y,z)\in\mathbb{R}^3$ which maps $\partial\mathcal{D}$ onto $S\cap\{z=0\}$. The coordinates y and z are smooth on $\mathcal{D}$. We denote $x_u, x_v, x_{uu}, x_{uv}, x_{vv}$ to be the partial derivatives of x with respect to u and v. The same notation is used for the partial derivatives of y with respect to u and v. Moreover, we have the following properties of the derivatives of the variable x:

$$x_u \sim \vartheta^{1-p},\ |x_{uv}|\le C\,\vartheta^{1-p} \quad\text{and}\quad x_{uu}\sim\vartheta^{2-p}$$

where ϑ behaves like distance to the boundary as in Section 5.
Let $T=(T_1,T_2,T_3)$ be smooth vector field transverse to S. Let $\eta>0$ be sufficiently small. We define the change of coordinates $\Phi:\mathcal{D}\times[-\eta,\eta]\to\mathbb{R}^3$ by

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \Phi\begin{pmatrix} u\\ v\\ w\end{pmatrix} = S(u,v)+wT(u,v) \tag{6.1}$$

or more explicitly

$$x = S_1(u,v)+wT_1(u,v) \tag{6.2}$$

$$y = S_2(u,v)+wT_2(u,v) \tag{6.3}$$

$$z = S_3(u,v)+wT_3(u,v) \tag{6.4}$$

Let $\delta > 0$ be small number, such that

$$T_1(u,v) = 0 \qquad \text{on } \mathcal{D} \setminus \mathcal{D}_{1-\delta}$$

denoting, as above, the transverse vector field to the surface $\mathcal{S}$. Notice that by choosing the smooth surface sufficiently close to the surface $z = h(x,y)$, we can make δ to depend only on the constant λ where λ depends on the initial non-degeneracy conditions on the surface Σ.

We write the first and second derivatives of z with respect to x, y and t in terms of the first and second derivatives of w with respect to u, v and t.
If $z = h_0(x,y)$ then we compute the first and second partial derivatives of z with respect to x and y in terms of $w = l(u,v)$ seen as functions of u and v.

Let A be the Jacobian matrix relative to the transformation of coordinates:

$$A = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let ∇z, ∇u, ∇v be, respectively the gradients of z, u and v:

$$\nabla z = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \nabla v = \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We denote by D^2u and D^2v the following matrices:

$$D^2u = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 u}{\partial x \partial y} \\ \dfrac{\partial^2 u}{\partial x \partial y} & \dfrac{\partial^2 u}{\partial y^2} \end{pmatrix} = \begin{pmatrix} a_1 & c_1 \\ c_1 & c_2 \end{pmatrix}$$

$$D^2v = \begin{pmatrix} \dfrac{\partial^2 v}{\partial x^2} & \dfrac{\partial^2 v}{\partial x \partial y} \\ \dfrac{\partial^2 v}{\partial x \partial y} & \dfrac{\partial^2 v}{\partial y^2} \end{pmatrix} = \begin{pmatrix} b_1 & d_1 \\ d_1 & d_2 \end{pmatrix}$$

Based on the above, we define $a_2 := c_1$, $b_2 := d_1$ and denote $e_1 = (1,0)$ $e_2 = (0,1)$ to be the basis vectors. Let A^{-1} be the inverse matrix of A:

$$A^{-1} := \begin{pmatrix} x_u, & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Next, we introduce the matrices B_1 and B_2 which denote, respectively, the derivative of the inverse matrix of A and A^{-1} with respect to x and y; $B_1 = \frac{\partial A^{-1}}{\partial x}$, $B_2 = \frac{\partial A^{-1}}{\partial y}$ which can be computed as:

$$B_1 = \begin{pmatrix} a\,x_{11} + b\,x_{12} & a\,y_{11} + b\,y_{12} \\ a\,x_{12} + b\,x_{22} & a\,y_{12} + b\,y_{22} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} c\,x_{11} + d\,x_{12} & c\,y_{11} + d\,y_{12} \\ c\,x_{12} + d\,x_{22} & c\,y_{12} + d\,y_{22} \end{pmatrix}$$

Note that we are using the following notation:

$$x_{11} = x_{uu},\, x_{12} = x_{uv},\, x_{22} = x_{vv},\, y_{11} = y_{uu},\, y_{12} = y_{uv},\, y_{22} = y_{vv}.$$

The coefficients of the matrices D^2u and D^2v are evaluated as follows:

$$\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = -A \cdot B_1 \cdot \nabla u; \quad \begin{pmatrix} b_1 \\ d_1 \end{pmatrix} = -A \cdot B_1 \cdot \nabla v$$

$$\begin{pmatrix} a_2 \\ c_2 \end{pmatrix} = -A \cdot B_2 \cdot \nabla u; \quad \begin{pmatrix} b_2 \\ c_2 \end{pmatrix} = -A \cdot B_2 \cdot \nabla v$$

Since: $\nabla z = A \cdot \begin{pmatrix} z_u \\ z_v \end{pmatrix} + z_w \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix}$, we obtain that:

$$\frac{\partial}{\partial x} \nabla z = A_x \cdot \begin{pmatrix} z_u \\ z_v \end{pmatrix} + A \cdot \begin{pmatrix} a\,z_{uu} + b\,z_{uv} + z_{uw}\frac{\partial w}{\partial x} \\ a\,z_{uv} + b\,z_{vv} + z_{vw}\frac{\partial w}{\partial x} \end{pmatrix} + (a\,z_{uw} + b\,z_{vw}) \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix} + z_w \begin{pmatrix} \frac{\partial^2 w}{\partial^2 x} \\ \frac{\partial^2 w}{\partial x \partial y} \end{pmatrix}.$$

$$\frac{\partial}{\partial y} \nabla z = A_y \cdot \begin{pmatrix} z_u \\ z_v \end{pmatrix} + A \cdot \begin{pmatrix} c\,z_{uu} + d\,z_{uv} + z_{uw}\frac{\partial w}{\partial y} \\ c\,z_{uv} + d\,z_{vv} + z_{vw}\frac{\partial w}{\partial y} \end{pmatrix} + (c\,z_{uw} + d\,z_{vw}) \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix} + z_w \begin{pmatrix} \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial^2 y} \end{pmatrix}.$$

The gradient of the function w, as well as its partial derivatives, can be expressed by using the matrix A:

$$\begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} a\,w_u + b\,w_v \\ c\,w_u + d\,w_v \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial^2 w}{\partial^2 x} \\ \frac{\partial^2 w}{\partial x \partial y} \end{pmatrix} = \begin{pmatrix} a_1\,w_u + a\,(a\,w_{uu} + b\,w_{uv}) + b_1\,w_v + b(a\,w_{uv} + b\,w_{vv}) \\ c_1\,w_u + c(a\,w_{uu} + b\,w_{uv}) + d_1\,w_v + d(a\,w_{uv} + b\,w_{vv}) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial^2 y} \end{pmatrix} = \begin{pmatrix} a_2\, w_u + a\,(c\, w_{uu} + d\, w_{uv}) + b_2\, w_v + b(c\, w_{uv} + d\, w_{vv}) \\ c_2\, w_u + c(c\, w_{uu} + d\, w_{uv}) + d_2\, w_v + d(c\, w_{uv} + d\, w_{vv}) \end{pmatrix}.$$

After perfoming all the substitutions above we get:

$$\frac{\partial^2 z}{\partial^2 x} = A^1_{11}\, w_{11} + A^1_{12}\, w_{12} + A^1_{22}\, w_{22} + B^1_1\, w_1 + B^1_2\, w_2 + B^1_{12}\, w_1\, w_2 + B^1_{11}\, w_1^2 + B^1_{22}\, w_2^2 + C_1$$

$$\frac{\partial^2 z}{\partial^2 y} = A^2_{11}\, w_{11} + A^2_{12}\, w_{12} + A^2_{22}\, w_{22} + B^2_1\, w_1 + B^2_2\, w_2 + B^2_{12}\, w_1\, w_2 + B^2_{11}\, w_1^2 + B^2_{22}\, w_2^2 + C_2$$

$$\frac{\partial^2 z}{\partial x \partial y} = A^\circ_{11}\, w_{11} + A^\circ_{12}\, w_{12} + A^\circ_{22}\, w_{22} + B^\circ_1\, w_1 + B^\circ_2\, w_2 + B^\circ_{12}\, w_1\, w_2 + B^\circ_{11}\, w_1^2 + B^\circ_{22}\, w_2^2 + C_\circ$$

where the coefficients $A^k_{i,j}$, B^j_i, C^i with $k = 0, 1, 2$, $i, j = 1, 2$ are defined as follows:

$$\begin{aligned}
A^1_{11} : &= -a^2\,(-z_w + b\,y_w(z_u + w_1\,z_w) + d\,y_w(z_v + w_2 z_w)) \\
A^1_{22} : &= -b^2\,(-z_w + b\,y_w(z_u + w_1\,z_w) + d\,y_w(z_v + w_2 z_w)) \\
A^1_{12} : &= -2\,a\,b\,(-z_w + b\,y_w(z_u + w_1\,z_w) + d\,y_w(z_v + w_2 z_w)) \\
A^2_{11} : &= c^2(z_w - d\,y_w(z_v + w_2\,z_w)) \\
A^2_{22} : &= d^2(z_w - d\,y_w(z_v + w_2\,z_w)) \\
A^2_{12} : &= -2cd(-z_w + d\,y_w(z_v + w_2 z_w)) \\
A^0_{11} : &= -ac(-z_w + by_w(z_u + w_1 z_w) + dy_w(z_v + w_2 z_w)) \\
A^0_{22} : &= -bd(-z_w + by_w(z_u + w_1 z_w) + dy_w(z_v + w_2 z_w)) \\
A^0_{12} : &= -b(2acy_{uw} + bcy_{vw} + ady_{vw})z_w \\
B^0_{11} : &= -b(2acy_{uw} + bcy_{vw} + ady_{vw})z_w \\
B^0_{22} : &= -d(bcy_{uw} + ady_{uw} + 2bdy_{vw})z_w \\
B^0_{12} : &= -((b^2c + a(b + 2c)d)y_{uw} + d(b(2b + c) + ad)y_{vw})z_w \\
B^0_1 : &= -a^2(cx_{uu} + dx_{uv})z_w - b(bcy_{vw}z_u + cdy_{vw}z_v - cz_{vw} + bcy_{uv}z_w + \\
&\quad bdy_{vv}z_w) - a(-2cz_{uw} + 2cdy_{uw}z_v + d^2y_{vw}z_v - dz_{vw} + b(2cy_{uw}z_u + dy_{vw}z_u + \\
&\quad c(x_{uv} + y_{uu})z_w + d(x_{uv} + y_{uv})z_w)) \\
B^0_2 : &= -b^2(cy_{uw} + 2dy_{vw})z_u - b(-cz_{uw} + d(ay_{uw}z_u + cy_{uw}z_v + 2dy_{vw}z_v - 2z_{vw}) + \\
&\quad (c(c + d)x_{uv} + cdy_{uv} + d^2y_{vv})z_w) - a(c^2x_{uu}z_w + d(-z_{uw} + c(x_{uv} + y_{uu})z_w + d(y_{uw}z_v + y_{uv}z_w))) \\
C^0 : &= -a^2(cx_{uu} + dx_{uv})z_u - a(b(c(x_{uv} + y_u u) + d(x_{uv} + y_{uv}))z_u - cz_{uu} + c^2x_{uu}z_v + \\
&\quad cd(x_{uv} + y_{uu})z_v + d(-z_{uv} + dy_{uv}z_v)) - b(b(cy_{uv} + dy_{vv})z_u - cz_{uv} + c^2x_{uv}z_v + c \\
&\quad d(x_{uv} + y_{uv})z_v + d(dy_{vv}z_v - z_{vv}))
\end{aligned}$$

$$
\begin{aligned}
B^1_{11}: &= -2ab(ay_{uw}+by_{vw})\\
B^1_{22}: &= -2bd(ay_{uw}+by_{vw})\\
B^1_{12}: &= -2(b^2+ad)(ay_{uw}+by_{vw})\,z_w\\
B^2_{11}: &= 0\\
B^2_{22}: &= -2\,d^2(c\,y_{uw}+d\,y_{vw})\,z_w\\
B^2_{12}: &= -2cd(cy_{uw}+dy_{vw})z_w\\
B^1_1: &= -2a(a(b\,y_{uw}\,z_u-z_{uw}+d\,y_{uw}z_v)+b(b\,y_{vw}\,z_u+d\,y_{vw}\,z_v-z_{vw}))-\\
&\quad (a^3\,x_{uu}+a^2b(2\,x_{uv}+y_{uu})+ab^2(x_{uv}+2\,y_{uv})+b^3y_{vv})z_w\\
B^1_2: &= -a^2(cx_{uu}+d\;y_{uu})z_w-2ab(by_{uw}z_u-z_{uw}+dy_{uw}z_v+cx_{uv}z_w+dy_{uv}z_w)-\\
&\quad b^2(2by_{vw}z_u+2dy_{vw}z_v-2z_{vw}+cx_{uv}z_w+dy_{vv}z_w)\\
C_1: &= -a^3x_{uu}z_u-a^2(b(2x_{uv}+y_{uu})z_u-z_{uu}+cx_{uu}z_v+dy_{uu}z_v)-\\
&\quad ab(b(x_{uv}+2y_{uv})z_u-2z_{uv}+2(cx_{uv}+dy_{uv})z_v)-b^2(by_{vv}z_u+cx_{uv}z_v+dy_{vv}z_v-z_{vv})\\
B^2_1: &= 2c((c(z_{uw}-dy_{uw}z_v)+d(-dy_{vw}z_v+z_{vw}))-ad^2x_{uv}z_w\\
B^2_2: &= -2d(-cz_{uw}+cdy_{uw}z_v+d^2y_{vw}z_v-dz_{vw})-(c^3x_{uu}+c^2\\
&\quad d(2x_{uv}+y_{uu})+cd^2(x_{uv}+2y_{uv})+d^3y_{vv})z_w\\
C_2: &= -ad^2x_{uv}z_u-c^3x_{uu}z_v+c^2(z_{uu}-d(2x_{uv}+y_{uu})z_v)+\\
&\quad cd(2z_{uv}-d(x_{uv}+2y_{uv})z_v)+d^2(-dy_{vv}z_v+z_{vv})
\end{aligned}
$$

Evolution of w_t. The evolution equation of w is obtained by looking at the evolution equation of $z=f(x,y,t)$. Namely,

$$\frac{\partial w}{\partial t}=\frac{\frac{\partial z}{\partial t}}{z_y\,y_w-z_w}$$

where the function z satisfies the non linear PDE:

$$z_t=\frac{z_{xx}z_{yy}-z_{xy}^2}{(1+z_y^2)z_{xx}-2\,z_xz_yz_{xy}-(1+z_x^2)z_{yy}} \tag{6.5}$$

Hence, by replacing the partial derivatives of z in terms of the derivatives of w, we find the linearization L to the above equation.

Linearization. Let $\tilde w$ be a point close to the initial data w_0. Then, the linearization L is given by

$$L(\tilde w)=a_{11}\,\tilde w_{11}+2\,a_{12}\,\tilde w_{12}+a_{22}\,\tilde w_{22}+b_1\,\tilde w_1+b_2\,\tilde w_2+c\,\tilde w+d$$

where the coefficients of the operator L behave as follows:

$$a_{11} \approx \frac{w_1^2}{w_{11}^2}\, g_{11};\ a_{12} \approx \frac{w_{11}\, w_1}{w_{11}^2}\, g_{12}$$
$$a_{11} \approx \frac{w_{11}^2}{w_{11}^2}\, g_{22};\ b_1 \approx \frac{w_{11}\, w_1}{w_{11}^2}\, h_1$$
$$b_2 \approx \frac{w_{11}^2}{w_{11}^2}\, h_2$$

where the functions g_{ij} and h_i, $i, j = 1, 2$ belong to the space C_s^α with particularly in g_{22} and b_2 in $C_s^{\alpha,p}$ such that the coefficients of the operator L satisfy the same conditions of the operator of Theorem 4.

We introduce the following definition to simplify the notation in the next theorem. We say that Σ_0 is of Hölder class $C_s^{k+2+\alpha,p}$ if the function $w(t)$ belongs the space $C_s^{k+2+\alpha,p}$. We can then state:

Theorem 6. *Assume that for some nonnegative integer k and some number α in $0 < \alpha < 1$, the initial surface Σ_0 belongs to the class $C_s^{k+2+\alpha,p}$ and satisfies the non-degeneracy condition ($\star$) . Then, under the coordinate change Φ, the (HMCF) with initial data the surface Σ_0 converts into the initial value problem*

$$\begin{cases} Mw = 0 & \text{on } \mathcal{D} \times [0, T] \\ w = w_0 & \text{at} \quad t = 0 \end{cases}$$

with $w^0 \in C_s^{k+2+\alpha,p}(\mathcal{D})$ and

$$Mw = w_t - F(t, u, v, w, Dw, D^2 w)$$

satisfying the hypotheses of Theorem 5.

As an immediate Corollary of Theorem 6 and Theorem 5 we obtain the following existence result.

Theorem 6.1. *Under the same assumptions as in Theorem 6, there exists a number $\tau_k > 0$ for which the (HMCF) with initial data the surface Σ admits a solution Σ_t on $0 \le t \le \tau_k$. Moreover, under the coordinate change Φ the strictly convex part $\Sigma_2(t)$ of Σ_t is converted to a function $w(t)$ which belongs to the Hölder class $C_s^{k+2+\alpha,p}(Q_k)$, on $Q_k = \mathcal{D} \times [0, \tau_k]$.*

7. The proof of the Main Theorem

In this section we will give the proof of the Main Theorem stated in Section 2. We will actually prove the following stronger result, where we relax the regularity assumptions on the initial surface.

Theorem 7. *Assume that the strictly convex part Σ_2 of the initial surface Σ belongs to the class $C_s^{2+\alpha,p}$, for some numbers p in $0 < p < 1$, $\alpha \le p$ and satisfies the non-degeneracy condition $(\star)$. Then, the HMCF*

$$\frac{\partial P}{\partial t} = \frac{K}{H}\ \vec{N} \qquad\qquad t \in [0,T]$$

with initial data the surface Σ admits a solution Σ_t which is smooth up to the interface, for $0 < t \le T$. In particular the interface Γ_t is a smooth curve for every $0 < t \le T$ and it moves by the Curve Shortening Flow.

Remark. It can be easily checked that if the initial surface satisfies the conditions of the Main Theorem, then it will satisfy the weaker conditions of Theorem 7.

Proof. Assume that the strictly convex part Σ_2 of the initial surface Σ belongs to the class $C_s^{2+\alpha,p}$, for some numbers p in $0 < p < 1$, $0 < \alpha \le 1$ and satisfies condition $(\star)$, then we have proven existence for the *HMCF* in Theorem 6.
From Theorem 6 we have that $w \in C_s^{k+2+\alpha,p}(\mathcal{D}\times(0,T])$, for all nonnegative integers k. In particular this implies that for all integers k we have $w \in C^{k,p}(\mathcal{D}\times(0,T])$, for all τ in $0 < \tau < T$. It follows that w is $C^{\infty,p}$ smooth up to the boundary of $\mathcal{D}$. Going back to the original coordinates, we conclude that the strictly convex part of the surface Σ_t, $0 < t \le T$ is smooth up to $z = 0$ and that the interface $\Gamma(t)$ is smooth.

□

8. Comparison principle

In this final section we will give the proof of the comparison principle for the (HMCF) and an observation regarding viscocity solutions.

Proposition 8. (Comparison principle) *Let Σ_0 be a surface of class $C_s^{2+\alpha,p}$ that satisfies condition $(\star)$, and let Σ^+ be a smooth, strictly convex surface containing Σ_0 at time $t = 0$, then the surface Σ_t obtained by evolving Σ_0 by the (HMCF), is contained in the surface Σ_t^+ obtained by evolving Σ_0^+ by the (HMCF) up to the time of existence of Σ_t. Analogously, if Σ_0 contains a smooth, strictly convex surface Σ^- at time $t = 0$, then the surface Σ_t contains the surface Σ_t^- obtained by evolving Σ_0^- by the (HMCF) up to the time of existence of Σ_t.*

Proof. We begin by observing that by the the classical maximum principle the surfaces Σ_t and Σ_t^+ cannot touch were they are both strictly convex. Suppose that there exists a time $\bar{t}$ where they first touch at a point $\bar{P}$, then this obviously cannot happen in the interior of the flat side, thus $\bar{P}$ has to belong to the boundary of the flat side. Suppose Σ_t has the flat side on the $x = 0$ plane, then the tangent to the surface at the point $\bar{P}$ would be parallel to $x = 0$ plane. This is because of the particular shape of the surface. Now if the two regions are touched, then because they are of class C^1 we would have that the tangent to $\Sigma^+_{\bar{t}}$ at $\bar{P}$ would be parallel to the $x = 0$ plane. But $\Sigma^+_{\bar{t}}$ is strictly convex, hence this would imply that a part of $\Sigma^+_{\bar{t}}$ is inside $\Sigma_{\bar{t}}$. This leads to a contradiction, and, therefore, the two regions never touch. The second part of the proof is straightforward since if Σ_0 contains a smooth, strictly convex surface Σ^- at time $t = 0$, then the two surfaces cannot touch at the flat side of Σ_t because the flat side does not move in its normal direction. Once again, by the classical maximum principle for parabolic equations the two surfaces cannot touch where they are strictly convex either. This observation concludes the proof of the proposition. □

Corollary 9. (Viscosity solutions) *Given an initial convex Σ surface of class $C^{1,\alpha}$ with a flat side as in Main Theorem, then the solution of the HMCF, Σ_t, found in the Main Theorem is a viscosity solution which converges uniformly to the initial surface Σ as $t \to 0$. Moreover, this solution is unique.*

Proof. It is a consequence of the comparison principle. □

9. Constant rate of decrease for the area

Finally we observe the following property of surfaces evolving by the *Harmonic Mean Curvature flow.*

Proposition 10. (Constant rate of decrease for the Area) *Let A be the surface area of a smooth, closed, strictly convex surface Σ evolving by HMCF. Then, we have*

$$\frac{dA}{dt} = -4\pi \tag{9.1}$$

Proof. For a strictly convex surface evolving with a certain speed v in the direction of the inward normal vector $\vec{N}$ by the equation $\frac{dP}{dt} = v\ \vec{N}$ the area decreases by the First Variation Formula. This means that:

$$\frac{dA}{dt} = -\int_\Sigma v\, H dp \tag{9.2}$$

where H indicates the mean curvature. For more details about how to derive this general formula see [5]. Thus, for the *HMCF*, $v = K/H$, which implies by the Gauss-Bonnet formula that the area shrinks at a constant rate:

$$\frac{dA}{dt} = -\int_\Sigma v\, H dp = -\int_\Sigma K dp = -4\pi$$

□

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