

ON THE EVOLUTION OF CONVEX HYPERSURFACES BY THE Q_k FLOW

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ABSTRACT. We prove the existence and uniqueness of a $C^{1,1}$ solution of the Q_k flow in the viscosity sense for compact convex hypersurfaces Σ_t embedded in \mathbb{R}^{n+1} ($n \geq 2$). The solution exists up to the time $T < \infty$ at which the enclosed volume becomes zero. In particular, for compact convex hypersurfaces with flat sides we show that, under a certain non-degeneracy initial condition, the interface separating the flat from the strictly convex side, becomes smooth, and it moves by the Q_{k-1} flow at least for a short time.

1. INTRODUCTION

We consider, in this paper, the evolution of a compact convex hypersurface Σ_t embedded in \mathbb{R}^{n+1} ($n \geq 2$) by the Q_k flow for $1 \leq k \leq n$, namely the equation

$$\frac{\partial \mathbf{P}}{\partial t} = -Q_k \nu$$

where each point \mathbf{P} of the surface Σ_t moves in the direction of its inner normal vector ν by a speed which is equal to the quotient

$$Q_k(\lambda) = \frac{S_k^n(\lambda)}{S_{k-1}^n(\lambda)}$$

of successive elementary symmetric polynomials of the principal curvatures $\lambda = (\lambda_1, \dots, \lambda_n)$ of Σ_t . We recall that

$$S_k^n(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Given a compact convex hypersurface Σ_0 embedded in \mathbb{R}^{n+1} ($n \geq 2$), we let $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be the immersion defining Σ_0 . We look for one parameter family

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of immersions $F : M^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$ satisfying

$$(*_k) \quad \begin{cases} \frac{\partial}{\partial t} F(p, t) = -Q_k(p, t) \cdot \nu(p, t), & \forall p \in M^n, t \geq 0, \\ F(\cdot, 0) = F_0 \end{cases}$$

where ν is the outer unit normal and $Q_k(\lambda)$ is a quotient of successive elementary symmetric polynomials of the principal curvatures of $\Sigma_t = F(M^n, t)$.

In [1] Andrews proved that for any strictly convex hypersurface in \mathbb{R}^{n+1} the solution to $(*_k)$ exists up to some finite time T at which it shrinks to a point in an asymptotically spherical manner. In [9] Dieter considered smooth, convex hypersurfaces in \mathbb{R}^{n+1} with $S_{k-1}^n(\lambda) > 0$. Using that condition she constructed cylindrically symmetric barriers to prove that instantaneously at time $t > 0$ the speed has a uniform lower bound from below, and therefore the flow becomes strictly parabolic.

In this article we will consider convex hypersurfaces, with no assumption on $S_{k-1}^n(\lambda)$. In the first part of the paper we will establish the existence and uniqueness of a $C^{1,1}$ solution of $(*_k)$ in the viscosity sense. We give the following definition of a viscosity solution suggested by Andrews in [2].

Definition 1.1. *A family of convex regions $\{\Omega_t\}_{0 < t < T}$ is called a viscosity solution of $(*_k)$ if the following holds. For any smooth, strictly convex hypersurface N_0 contained in Ω_{t_0} , for some $t_0 \in (0, T)$, the hypersurfaces N_t given by solving $(*_k)$ are contained in Ω_{t_0+t} , for all $t \in [0, T - t_0)$ in the time interval of existence of N_t . Furthermore, for any smooth, strictly convex hypersurface N_0 which encloses Ω_{t_0} , for some $t_0 \in (0, T)$, the hypersurfaces N_t enclose Ω_{t_0+t} , for all $t \in [0, T - t_0)$.*

Our first result states as follows:

Theorem 1.2. *Let Σ_0 be a compact convex hypersurface in \mathbb{R}^{n+1} which is of class $C^{1,1}$. Then, there exists a unique viscosity solution of the $(*_k)$ flow which is of class $C^{1,1}$. The solution exists up to the time $T < \infty$ at which the enclosed volume becomes zero.*

In the second part of the paper we will consider the special case where the initial surface Σ_0 has flat sides. We will assume for simplicity that Σ_0 has only one flat side, namely $\Sigma_0 = \Sigma_0^1 \cup \Sigma_0^2$ with Σ_0^1 flat (i.e. it lies on a hyperplane in \mathbb{R}^{n+1}) and Σ_0^2 strictly convex. Since the equation is invariant under rotation, we may assume

that Σ_0^1 lies on the $x_{n+1} = 0$ plane and that Σ_0^2 lies above this plane. Then, the lower part of the surface Σ_0 can be written as the graph of a function

$$x_{n+1} = u(x_1, \dots, x_n)$$

over a compact domain $\Omega \subset \mathbb{R}^n$ containing the initial flat side Σ_0^1 .

We define

$$g := \sqrt{u}$$

and we call g the *pressure function*. Let Γ_0 denote the boundary of the flat side Σ_0^1 . Our main assumption on the initial surface Σ_0 is that it is of class $C^{1,1}$, the function g is C^2 up to the boundary of the flat side and it satisfies the following non-degeneracy condition, which we call *non-degeneracy condition* (\star) :

$$(\star) \quad |Dg| \geq \lambda \quad \text{and} \quad (g_{ij}) \geq \lambda, \quad \text{on } \Gamma_0$$

for some number $\lambda > 0$.

For each $1 \leq i, j \leq n-1$, g_{ij} denote the second order derivatives in the directions given by the vectors τ_i and τ_j . For each $1 \leq i \leq n-1$, τ_i is defined so that the set $\text{Span}[\tau_1, \dots, \tau_{n-1}]$ is parallel to the tangent hyperplane to the level sets of g .

Definition 1.3. We define \mathfrak{S} to be the class of convex compact hypersurfaces Σ in \mathbb{R}^{n+1} so that $\Sigma = \Sigma^1 \cup \Sigma^2$, where Σ^1 is a surface contained in the hyperplane $x_{n+1} = 0$ with smooth boundary Γ , and Σ^2 is a strictly convex surface, smooth up to its boundary Γ which lies above the hyperplane $x_{n+1} = 0$.

Remark 1.4. Any initial surface Σ_0 in the class \mathfrak{S} is in particular a $C^{1,1}$ surface. Hence, by Theorem 1.2, there exists a unique $C^{1,1}$ solution Σ_t of $(*_k)$ with initial data Σ_0 .

We will assume that $x_{n+1} = u(x_1, \dots, x_n, t)$ defines the hypersurface Σ_t near the hyperplane $x_{n+1} = 0$, with $0 \leq t \leq \tau$ for some short time $\tau > 0$. We will set

$$g(\cdot, t) = \sqrt{u(\cdot, t)}.$$

Our goal is to show the following result:

Theorem 1.5. Assume that at time $t = 0$, Σ_0 is a weakly convex compact hypersurface in \mathbb{R}^{n+1} which belongs to the class \mathfrak{S} so that the pressure function $g = \sqrt{u}$ is smooth up to the interface Γ_0 and it satisfies the condition (\star) . Let Σ_t be the unique viscosity solution of $(*_k)$ for $2 \leq k \leq n$ with initial data Σ_0 . Then, there

exists a time $\tau > 0$ such that the pressure function $g(\cdot, t) = \sqrt{u(\cdot, t)}$ is smooth up to the interface $x_{n+1} = 0$ and satisfies condition (\star) for all $t \in [0, \tau)$. In particular, the interface Γ_t between the flat side and the strictly convex side is a smooth hypersurface for all t in $0 < t \leq \tau$ and it moves by the $(*_k)$ flow.

Remark 1.6. In the case of a two-dimensional surface in \mathbb{R}^3 the flow $(*_k)$ becomes the well studied harmonic mean curvature flow. In this case, Theorem 1.5 was established in [4]. Following the result in [4] one may consider a pressure function $g = u^p$, for any number $p \in (0, 1)$, and prove the short time existence of a solution to the $(*_k)$ flow which is of class $C^{m, \gamma}$ (with m, γ depending on p) so that the pressure function g is still smooth up to the interface and the interface moves by the Q_{k-1} flow. The fact that the solution Σ_t remains in the class $C^{m, \gamma}$, for $t > 0$, distinguishes this flow from other, previously studied, degenerate free-boundary problems (such as the Gauss curvature flow with flat sides, the porous medium equation and the evolution p-laplacian equation) in which the regularity of the solution for $t > 0$ does not depend on the regularity of the initial data.

The paper is organized as follows: in section 2.2 we will present some a priori estimates for strictly convex surfaces, most of which have been shown by Andrews in [1]. In section 2.3 we will prove the existence of a solution to $(*_k)$, as stated in Theorem 1.2. The uniqueness of solutions will be shown in section 2.4. Section 3 will be devoted to the proof of Theorem 1.5.

2. PART I: THE EXISTENCE AND UNIQUENESS OF A $C^{1,1}$ SOLUTION

2.1. Notation and evolution equations. Since the right hand side of $(*_k)$ can be viewed as a function of the second fundamental form matrix A , a direct computation shows that its linearization is given by

$$\mathcal{L}(u) = \frac{\partial Q_k}{\partial h_i^i} \nabla_i \nabla_l u = a^{il} \nabla_i \nabla_l u$$

with

$$(2.1) \quad a^{il} = \frac{\partial Q_k}{\partial h_i^i}.$$

Notice that if we compute a^{il} in geodesic coordinates around the point (at which the matrix A is diagonal) we get

$$a^{ii} = \frac{\partial Q_k}{\partial \lambda_i}$$

and other elements being zero.

Also, we have the evolution equations of the induced metric g_{ij} ,

$$(2.2) \quad \frac{\partial g_{ij}}{\partial t} = -2Q_k h_{ij}$$

of the mean curvature H ,

$$(2.3) \quad \frac{\partial H}{\partial t} = \mathcal{L}H + \frac{\partial^2 Q_k}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_i h_m^l + \frac{\partial Q_k}{\partial h_m^l} h_p^l h_m^p H$$

and of the speed Q_k ,

$$(2.4) \quad \frac{\partial Q_k}{\partial t} = \mathcal{L}Q_k + \frac{\partial}{\partial h_j^i} Q_k h^{il} h_{lj} Q_k.$$

2.2. Apriori estimates for strictly convex hypersurfaces. The main result in this section relies on the work of Andrews in [1]. Before we state it, let's prove the following lemma.

Lemma 2.1. *Let Σ be a convex hypersurface (not necessarily strictly convex) such that it contains a ball $B_\rho(0)$, centered at the origin, of radius ρ . Then, for every $q \in \Sigma$, we have*

$$\langle q, \nu \rangle \geq \rho$$

where $\nu(q)$ denotes the outer unit normal to Σ at q .

Proof. Let Ω be the convex domain with Σ as boundary. Let $q \in \Sigma$, and $\nu(q)$ denote the outer unit normal to Σ at q . Then $\langle q, \nu(q) \rangle = \max_{x \in \Omega} \langle x, y \rangle$, where $y = \nu(q)$. This is greater than or equal to ρ since $B_\rho(0) \subset \Omega$. □

Theorem 2.2. *Let Σ_0 be a strictly convex hypersurface in \mathbb{R}^{n+1} and let Σ_t be a family of hypersurfaces evolving by $(*_k)$. Let $\tau > 0$ be such that a ball B_ρ , of radius ρ is contained in Σ_τ . Then, there exists a positive constant $C = C(\rho, \|\Sigma_0\|_{C^{1,1}})$ so that*

$$|A(t)| \leq C, \quad \forall 0 \leq t \leq \tau,$$

where $A(t)$ is the second fundamental form of Σ_t .

Proof. Assume with no loss of generality that the origin is enclosed by Σ_0 and

$$B_\rho(0) \text{ is contained in } \Sigma_\tau.$$

As in [8] we consider the quantity

$$\mathcal{F} = \langle F, \nu \rangle + 2tQ_k.$$

Since the speed Q_k is a homogeneous function of the principal curvatures of degree one, a similar computation as in [8] yields to

$$(2.5) \quad \frac{\partial}{\partial t} \langle F, \nu \rangle = \mathcal{L} \langle F, \nu \rangle - 2Q_k + \frac{\partial Q_k}{\partial h_j^i} h^{il} h_{lj} \langle F, \nu \rangle.$$

Hence

$$(2.6) \quad \frac{\partial}{\partial t} \mathcal{F} = \mathcal{L} \mathcal{F} + \frac{\partial Q_k}{\partial h_j^i} h^{il} h_{lj} \mathcal{F}.$$

As in [9] we have

$$(2.7) \quad \sum_{i=1}^n \frac{\partial Q_k(\lambda)}{\partial \lambda_i} \lambda_i^2 \geq \frac{k}{n-k+1} Q_k^2(\lambda) \geq 0.$$

This together with the maximum principle applied to (2.6) imply

$$\mathcal{F}_{\min}(t) \nearrow \text{ in } t,$$

and therefore by Lemma 2.1,

$$\mathcal{F}_{\min}(t) \geq \mathcal{F}_{\min}(0) = \langle F, \nu \rangle_{\min}(0) \geq \rho > 0, \quad \text{for all } t \geq 0.$$

The speed Q_k is concave and from the evolution equation for $\frac{H}{\mathcal{F}}$ one can easily deduce,

$$\frac{\partial}{\partial t} \left(\frac{H}{\mathcal{F}} \right) \leq \mathcal{L} \left(\frac{H}{\mathcal{F}} \right) + \frac{2}{\mathcal{F}} \frac{\partial Q_k}{\partial h_j^i} \nabla_j \mathcal{F} \nabla^i \left(\frac{H}{\mathcal{F}} \right).$$

By the maximum principle applied to the previous inequality we get

$$\sup_{\Sigma_t} \frac{H}{\mathcal{F}} \searrow \text{ in } t.$$

Since $|\langle F, \nu \rangle| \leq |F| \leq C$, uniformly in t , there are uniform constants C_1, C_2 so that

$$(2.8) \quad H(\cdot, t) \leq C_1 + C_2 Q_k.$$

Also, because our surface is convex, the second fundamental form is controlled by the mean curvature H . In addition, since we assume that the ball $B_\rho \subset \Sigma_\tau$, by Lemma 2.1

$$\langle F, \nu \rangle \geq 2\rho, \quad \text{for all } t \in [0, \tau].$$

As in [1] we consider the quantity

$$\frac{Q_k}{\langle F, \nu \rangle - \rho}.$$

The evolution equation for Q_k , (2.5) and (2.7) yield to:

$$\begin{aligned}
 \frac{\partial}{\partial t} \left(\frac{Q_k}{\langle F, \nu \rangle - \rho} \right) &= \mathcal{L} \left(\frac{Q_k}{\langle F, \nu \rangle - \rho} \right) + \frac{2}{\langle F, \nu \rangle - \rho} \frac{\partial Q_k}{\partial h_j^i} \nabla^i \left(\frac{Q_k}{\langle F, \nu \rangle - \rho} \right) \nabla_j \langle F, \nu \rangle \\
 &+ \frac{Q_k}{(\langle F, \nu \rangle - \rho)^2} \left(2Q_k - \rho \frac{\partial Q_k}{\partial h_j^i} h^{il} h_{lj} \right) \\
 &\leq \mathcal{L} \left(\frac{Q_k}{\langle F, \nu \rangle - \rho} \right) + \frac{2}{\langle F, \nu \rangle - \rho} \frac{\partial Q_k}{\partial h_j^i} \nabla^i \left(\frac{Q_k}{\langle F, \nu \rangle - \rho} \right) \nabla_j \langle F, \nu \rangle \\
 (2.9) \quad &+ \frac{Q_k^2}{(\langle F, \nu \rangle - \rho)^2} \left(2 - \rho \frac{k}{n-k+1} Q_k \right).
 \end{aligned}$$

If $2 - \rho \frac{k}{n-k+1} Q_k \geq 0$, this implies that $Q_k \leq \frac{2(n-k+1)}{k\rho}$ and because of (2.8) we are done. If $2 - \rho \frac{k}{n-k+1} Q_k < 0$, the maximum principle applied to (2.9) yields to

$$\frac{d}{dt} \left(\frac{Q_k}{\langle F, \nu \rangle - \rho} \right)_{\max} \searrow \text{ in } t$$

which, in particular implies that

$$Q_k(t) \leq C$$

for a constant C that depends on ρ and that $C^{1,1}$ norm of Σ_0 . \square

2.3. The existence of a viscosity solution to $(*_k)$. We will next show the following proposition which constitutes the existence part of Theorem 1.2.

Proposition 2.3. *Under the initial assumptions of Theorem 1.2, there exists a $C^{1,1}$ solution Σ_t to the $(*_k)$ flow up to some finite time T at which the enclosed volume shrinks to zero.*

Proof. We begin by approximating our initial convex hypersurface Σ_0 by a family of smooth strictly convex surfaces Σ_0^ϵ (for example by the mean curvature flow). We may assume that all Σ_0^ϵ are contained in Ω , approaching $\Sigma_0 = \partial\Omega_0$ in the Hausdorff distance as $\epsilon \rightarrow 0$ and that Σ_0^ϵ increase in ϵ . Let Σ_t^ϵ , $t \in [0, T_\epsilon)$, be the unique strictly solution of $(*_k)$ flow with initial data Σ_0^ϵ . Then, Σ_t^ϵ increases in ϵ . In addition, T_ϵ denotes the time at which the surface Σ_t^ϵ shrinks to a point, then T_ϵ increases in ϵ . Moreover, an easy application of the comparison principle shows that $T_\epsilon \leq T_R$, where T_R is the extinction times of a sphere S_R which can be placed outside of Σ_0 . Hence, the limit

$$T := \lim_{\epsilon \rightarrow 0} T_\epsilon$$

exists. Let $\tau < T$. Then there exist $\rho > 0$ and ϵ_0 so that $B_\rho \subset \Sigma^\epsilon(\tau)$, for all $\epsilon \leq \epsilon_0$. Theorem 2.2 implies the uniform bound

$$\|\Sigma_t^\epsilon\|_{C^{1,1}} \leq C(\tau, \|\Sigma\|_{C^{1,1}}), \quad \text{for all } t \in [0, \tau].$$

Since each Σ^ϵ is smooth, the previous estimate shows that

$$\|\Sigma_t^\epsilon\|_{C^2} \leq C(\tau, \|\Sigma\|_{C^{1,1}}), \quad t \in [0, \tau].$$

Also, since Σ_t^ϵ is increasing in ϵ , the Arzela-Ascoli theorem and the previous estimate imply there is a $C^{1,1}$ limit

$$\Sigma_t := \lim_{\epsilon \rightarrow 0} \Sigma_t^\epsilon$$

and that the convergence is in the $C^{1,1}$ norm.

We claim that Σ_t is a viscosity solution to $(*_k)$ in the sense of Definition 1.1. To see that, let $\Sigma'_0 \subset \Omega$ where $\partial\Omega = \Sigma_0$. Then Σ'_0 is enclosed by Σ_0^ϵ , for ϵ sufficiently small. By the comparison principle Σ'_t is enclosed by Σ_t^ϵ , for $t > 0$. Since Σ_t^ϵ increases in ϵ and converges to Σ_t , we have that Σ'_t is enclosed by Σ_t , for $t > 0$. The second condition in Definition 1.1 can be checked similarly.

We finally observe that the enclosed volume of Σ_t shrinks to zero, as $t \rightarrow T$. Indeed, assume otherwise. Then, there exists a sphere $S_{2\rho}(P)$, for some $\rho > 0$, which is enclosed by Σ_t , for all $t < T$. Fix a $t_0 < T$ to be chosen momentarily. Since $\Sigma_\epsilon^{t_0} \nearrow \Sigma_{t_0}$, as $\epsilon \rightarrow 0$, this means that there exists an $\epsilon = \epsilon(t_0)$ such that $S_\rho(P)$ is enclosed inside the surface $\Sigma_\epsilon^{t_0}$. The comparison principle then shows that the vanishing time T_ϵ of $\Sigma_\epsilon^{t_0}$ satisfies

$$T_\epsilon \geq t_0 + T_\rho$$

where T_ρ is the time at which the sphere $S_\rho(P)$ evolving by our $(*_k)$ flow shrinks to a point. On the other hand, since

$$T \geq T_\epsilon \geq t_0 + T_\rho$$

we will reach a contradiction provided that t_0 is chosen sufficiently close to T , depending only on ρ . This completes the proof of our proposition. \square

2.4. The uniqueness of a viscosity solution to $(*_k)$. In Proposition 2.3 we have constructed a viscosity solution to $(*_k)$. The question that arises is whether that solution is unique in the class of viscosity solutions defined by Definition 1.1. We will give a positive answer to this question in the following proposition.

Proposition 2.4. *If $F_1(\cdot, t), F_2(\cdot, t)$ are two viscosity solutions to $(*_k)$, then*

$$F_1 \equiv F_2.$$

Proof. Let Σ_0 be our initial surface and Σ_0^ϵ the strictly convex approximating surfaces considered in Proposition 2.3, which are all enclosed by Σ_0 . We may assume, without loss of generality, that all Σ_0^ϵ contain a ball $B_\rho(0)$ centered at the origin. Let us consider another strictly convex surface $\tilde{\Sigma}_0^{\epsilon\delta}$ defined as a dilation of Σ_0^ϵ , that is, if Σ_0^ϵ is defined by an immersion F_0^ϵ , then

$$\tilde{F}_0^{\epsilon\delta} = (1 + \delta) F_0^\epsilon$$

defines the surface $\tilde{\Sigma}_0^{\epsilon\delta}$.

Claim 2.5. *For every $\delta > 0$ there exists an $\epsilon = \epsilon(\delta)$ so that the surface $\tilde{\Sigma}_0^{\epsilon\delta}$ defined as above encloses the surface Σ_0 .*

Proof. Observe that

$$|\tilde{F}_0^{\epsilon\delta}| - |F_0| = |F_0^\epsilon| - |F_0| + \delta|F_0^\epsilon|$$

and that $\|F_0^\epsilon - F_0\| < \alpha(\epsilon)$, where $\alpha(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Since all Σ_0^ϵ contain the ball $B_\rho(0)$, we have

$$|\tilde{F}_0^{\epsilon\delta}| - |F_0| \geq -\alpha(\epsilon) + \delta\rho > 0, \quad \text{if } \alpha(\epsilon) < \delta\rho.$$

Hence, by choosing $\epsilon := \epsilon(\delta)$ sufficiently small so that $\alpha(\epsilon) < \delta\rho$, we guarantee that $\tilde{\Sigma}_0^{\epsilon\delta}$ encloses Σ_0 . \square

From now on fix $\delta > 0$, set $\epsilon = \epsilon(\delta)$ and consider the corresponding surfaces Σ^ϵ and $\tilde{\Sigma}_0^{\epsilon\delta}$. Then, $\tilde{\Sigma}_0^{\epsilon\delta}$ encloses Σ_0 and Σ_0 encloses Σ^ϵ . We write briefly that

$$\Sigma_0^\epsilon \prec \Sigma_0 \prec \tilde{\Sigma}_0^{\epsilon\delta}.$$

By Definition 1.1, we will have

$$(2.10) \quad \Sigma_t^\epsilon \prec \Sigma_t \prec \tilde{\Sigma}_t^{\epsilon\delta}, \quad \forall t < T_\epsilon$$

where T_ϵ denotes the extinction time of Σ_t^ϵ , since Σ_t is a viscosity solution.

Denote by $\Sigma_t^\epsilon, \tilde{\Sigma}_t^{\epsilon\delta}$ the smooth solutions of $(*_k)$ starting at $\Sigma_0^\epsilon, \tilde{\Sigma}_0^{\epsilon\delta}$ respectively (note that both solutions are unique because their initial data are strictly convex).

It is easy to see that $\tilde{\Sigma}_t^{\epsilon\delta}$ is given by the immersion

$$\tilde{F}^{\epsilon\delta}(\cdot, t) = (1 + \delta) F^\epsilon\left(\cdot, \frac{t}{(1 + \delta)^2}\right).$$

By above $\tilde{F}^{\epsilon\delta}(t(1+\delta)^2) = (1+\delta)F^\epsilon(t)$, and we compute

$$\begin{aligned} \frac{\partial}{\partial t} [\tilde{F}^{\epsilon\delta}(t(1+\delta)^2) - F^\epsilon(t)] &= Q_k^\epsilon \nu^\epsilon(t)(1+\delta) - Q_k^\epsilon \nu^\epsilon(t) \\ &= \delta Q_k^\epsilon \nu^\epsilon(t) \end{aligned}$$

which implies that

$$(2.11) \quad \frac{\partial}{\partial t} |\tilde{F}^{\epsilon\delta}(\cdot, t(1+\delta)^2) - F^\epsilon(\cdot, t)| \leq \delta Q_k^\epsilon(t)$$

where $Q_k^\epsilon(t)$ is the speed of the flow $(*_k)$ starting at the surface Σ_0^ϵ .

Denote, as before by T_ϵ the extinction time of Σ_t^ϵ and let $T = \lim_{\epsilon \rightarrow 0} T^\epsilon$. Take any $\tau < T$. Then, there exist $\delta_0 > 0$ and $\rho = \rho(\tau)$ so that

$$B_\rho \subset \text{Int}(\Sigma_t^\epsilon), \quad \forall \delta < \delta_0, \quad \forall t \leq \tau < T^\epsilon \leq T$$

for an $\epsilon = \epsilon(\delta)$ which is defined as above. By Theorem 2.2 it follows that the speed $\tilde{Q}_k^{\epsilon\delta}$ of $\tilde{\Sigma}_t^{\epsilon\delta}$ satisfies

$$(2.12) \quad |\tilde{Q}_k^{\epsilon\delta}(t)| \leq C(\tau, \|\Sigma\|_{C^{1,1}}), \quad \forall \delta \leq \delta_0, \quad \forall t \leq \tau.$$

Estimate (2.11) yields to

$$\begin{aligned} (2.13) \quad |\tilde{F}^{\epsilon\delta}(t(1+\delta)^2) - F^\epsilon(t)| &\leq |\tilde{F}_0^{\epsilon\delta} - F_0^\epsilon| + \delta C(\tau) \\ &\leq \delta (|F_0^\epsilon| + C(\tau)) \\ &\leq \tilde{C}(\tau) \delta \end{aligned}$$

Since $\tilde{F}^{\epsilon\delta}(t)$ solves the $(*_k)$ with and $\tilde{Q}_k^{\epsilon\delta}(t) = \frac{Q_k^\epsilon(t(1+\delta)^{-2})}{1+\delta}$, it follows from (2.12) that

$$\begin{aligned} (2.14) \quad |\tilde{F}^{\epsilon\delta}(t(1+\delta)^2) - \tilde{F}^{\epsilon\delta}(t)| &\leq C(\tau) \tau ((1+\delta)^2 - 1) + |\tilde{F}_0^{\epsilon\delta} - F_0^\epsilon| \\ &\leq (\tilde{C}(\tau) + |F_0^\epsilon|) \delta \leq \bar{C}(\tau) \delta \end{aligned}$$

Combining (2.13) and (2.14) yields to

$$(2.15) \quad |\tilde{F}^{\epsilon\delta}(t) - F^\epsilon(t)| \leq C(\tau) \delta, \quad \forall t \in [0, \tau], \quad \delta < \delta_0$$

for $\epsilon = \epsilon(\delta)$ defined as above. Here, $C(\tau)$ is a uniform constant that does not depend on δ . In particular, (2.15) implies that the viscosity solutions which are obtained as the limits of $\{F^\epsilon(t)\}$ and $\{\tilde{F}^{\epsilon\delta}(t)\}$, as $\delta \rightarrow 0$ (whose existence is justified by Proposition 2.3) are the same.

We will now conclude the proof of uniqueness. Let Σ_t^1 and Σ_t^2 be two families of viscosity solutions to $(*_k)$ starting at Σ_0 . Let $\{\Sigma^\epsilon\}$ and $\{\tilde{\Sigma}^{\epsilon\delta}\}$ be two families of approximations of Σ by strictly convex surfaces taken as above. It follows by

the Definition 1.1 of viscosity solutions, that both Σ_t^1 and Σ_t^2 satisfy (2.10). Hence, from (2.15) we have

$$|F^1(t) - F^2(t)| \leq |\tilde{F}^{\epsilon\delta}(t) - F^\epsilon(t)| \leq C(\tau)\delta, \quad \forall t < \tau.$$

Letting $\delta \rightarrow 0$ we obtain that $\Sigma_t^1 = \Sigma_t^2$ for all $t \in [0, \tau]$. Since $\tau < T$ was arbitrary, we conclude that $\Sigma_t^1 = \Sigma_t^2$, for all $t \in [0, T)$, which finishes our proof. \square

2.5. Discussion on strict convexity. In this section we will give some observations as to when a convex surface, which is not necessarily strictly convex, actually becomes strictly convex as soon as it moves away from the initial surface. We first observe that the speed Q_k is bounded from below away from zero at points of Σ_t which are away from the initial surface Σ_0 .

Proposition 2.6. *Assume that Σ_0 is a $C^{1,1}$ compact convex hypersurface embedded in \mathbb{R}^{n+1} and let Σ_t denote the unique $C^{1,1}$ solution of the $(*_k)$ flow with initial data Σ_0 which exists on $0 < t < T$. Assume that at some point $P \in \Sigma_{t_0}$, with $t_0 < T$, we have $d_P := \text{dist}(P, \Sigma_0) > 0$. Then, there exist positive constants $\delta > 0$, $\tau > 0$ and $c > 0$, depending only on d_P and the diameter of the initial surface Σ_0 , such that the speed $Q_k \geq c > 0$ at all points $Q \in \Sigma_t$ with $\text{dist}_{M^n}(P, Q) < \delta$ and $t \in [t_0 - \tau, t_0 + \tau]$.*

Proof. The idea of the proof is simple. For the given point P we will consider the quantity

$$\mathcal{F} := \langle F - P, \nu \rangle + 2t Q_k$$

which we already introduced in the proof of Theorem 2.2 where we also showed that its minimum is increasing in time. Since $B_{d_P/2}(P)$ is strictly contained in the initial surface Σ_0 , Lemma 2.1 implies that at time $t = 0$ we have $\mathcal{F}_{\min}(0) \geq c_0 > 0$ for some constant $c_0 > 0$ depending only on d_P and the diameter of Σ_0 . Hence,

$$\mathcal{F}(p, t_0) = 2t_0 Q_k(p, t_0) \geq c_0 > 0$$

implying that the speed Q_k of the surface is strictly positive near P (here we denote by $p \in M^n$ the point such that $F(p, t_0) = P$).

To make the above argument rigorous, we let Σ_0^i be a decreasing sequence of strictly convex surfaces which approximate Σ_0 , i.e. we have $\Sigma_0^i \searrow \Sigma_0$ in the $C^{1,1}$ norm. We denote by Σ_t^i the solution of the $(*_k)$ flow with initial data Σ_0^i and by T^i its vanishing time. Then, it follows from the proof of Theorem 1.2 that $\Sigma_t^i \searrow \Sigma_t$

in the $C^{1,1}$ norm for all $0 < t < T$. Pick points $P^i \in \Sigma_{t_0}^i$ such that $P^i \rightarrow P$ and choose i_0 sufficiently large so that

$$\text{dist}(P_i, \Sigma_0^i) \geq \frac{d_p}{2} := d > 0, \quad \forall i \geq i_0.$$

Fixing $i \geq i_0$ for the moment, we consider the quantity

$$\mathcal{F}^i(\cdot, t) = \langle F^i - P, \nu \rangle + 2t Q_k^i$$

with F^i and Q_k^i denoting the position function and speed of Σ_t^i respectively. Since, $B_d(P_i)$ is contained in the initial surface Σ_0^i , Lemma 2.1 implies that

$$\mathcal{F}_{\min}^i(0) \geq c_0 > 0$$

for some constant c_0 depending only on d and the diameter of Σ_0 . The maximum principle applied to the evolution of \mathcal{F}^i (as in the proof of Theorem 2.2) implies that

$$(2.16) \quad \mathcal{F}_{\min}^i(t) \geq c_0 > 0, \quad \forall t < T.$$

Denote by $p \in M^n$ the point at which $F(p, t_0) = P$. For the given time t_0 , choose $\tau > 0, \epsilon > 0$ sufficiently small so that

$$|F^i(q, t) - P| \leq \frac{c_0}{2}, \quad \forall t \in [t_0 - \tau, t_0 + \tau], \quad \text{dist}_{M^n}(p, q) < \epsilon$$

with c_0 the constant in (2.16). The a priori estimates in Theorem 2.2 imply that τ and ϵ can be chosen to be independent of i . It follows from (2.16) that

$$Q_k^i(q, t) \geq \frac{c_0}{4t} \geq c > 0, \quad \forall t \in [t_0 - \tau, t_0 + \tau], \quad \text{dist}_{M^n}(p, q) < \epsilon.$$

The proposition now follows from the observation that Σ_t^i have uniformly bounded $C^{1,1}$ norms after we pass to the limit. Passing to the limit yields

$$(2.17) \quad Q_k(q, t) \geq c > 0, \quad \forall t \in [t_0 - \tau, t_0 + \tau], \quad \text{dist}_{M^n}(p, q) < \epsilon.$$

□

Corollary 2.7. *Assume that Σ_0 is a $C^{1,1}$ compact convex hypersurface embedded in \mathbb{R}^{n+1} and let Σ_t denote the unique $C^{1,1}$ solution of the $(*_k)$ flow with initial data Σ_0 which exists on $0 < t < T$. If $\text{dist}(\Sigma_t, \Sigma_0) > 0$, then Σ_t is strictly convex.*

Proof. We combine Proposition 2.6 and a recent constant rank theorem by Bian and Guan in [3]. By the previous proposition $Q_k \geq C(t) > 0$ on Σ_t . This in particular implies that the equation $(*_k)$ is strictly parabolic and that the surface Σ_t is smooth. It then follows from the constant rank theorem in [3] that at any

given time $t > 0$ the rank of the second fundamental form of the surface Σ_t is constant. Since Σ_t is a smooth compact surface, there exists at least a point P at which Σ_t is strictly convex, which forces the whole surface to be strictly convex, finishing the proof. \square

We next observe that combining the previous corollary and the main result of Dieter in [9] we obtain the following:

Corollary 2.8. *Assume that Σ_0 is a $C^{1,1}$ compact convex hypersurface embedded in \mathbb{R}^{n+1} and let Σ_t denote the unique $C^{1,1}$ solution of the $(*_k)$ flow with initial data Σ_0 which exists on $0 < t < T$. If $S_{k-1}^n > 0$ uniformly on Σ_0 , then the solution Σ_t is strictly convex for all $t > 0$.*

Proof. The main result by Dieter in [9] shows that if $S_{k-1}^n > 0$ uniformly on Σ_0 , then the speed Q_k satisfies the bound $Q_k > C(t) > 0$ on Σ_t . Hence, as in the previous corollary Σ_t is strictly convex. \square

One may ask: does it follow from Proposition 2.6 that the surface Σ_t is strictly convex locally near points P which are away from the initial surface? The same proposition shows that the k largest principal curvatures $\lambda_1, \dots, \lambda_k$ are positive at points of Σ_t which are away from Σ_0 . However, some of the other principal curvatures may vanish. On the other hand, since the constant rank theorem in [3] is local, Proposition 2.6 implies that the rank of the second fundamental form of the surface Σ_t is constant on each connected component of $\Sigma_t \setminus \Sigma_0$. Hence, those connected components that contain at least one point at which the surface is strictly convex are indeed strictly convex. The question as to whether $\Sigma_t \setminus \Sigma_0$ is always strictly convex remains open for investigation. However, our discussion above leads to the following observation.

Corollary 2.9. *Assume that Σ_0 is a $C^{1,1}$ compact convex hypersurface embedded in \mathbb{R}^{n+1} and let Σ_t denote the unique $C^{1,1}$ solution of the $(*_n)$ flow with initial data Σ_0 which exists on $0 < t < T$. Assume that at some point $P \in \Sigma_{t_0}$, with $t_0 < T$, we have $d_P := \text{dist}(P, \Sigma_0) > 0$. Then, there exist positive constant $\delta > 0$, depending only on d_P and the initial surface Σ_0 , such that Σ_t is strictly convex at all points $Q \in \Sigma_t$ such that $\text{dist}(P, Q) < \delta$.*

Proof. Denote by λ_j , $j = 1, \dots, n$ the principal curvatures of the surface Σ_t and assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Hence,

$$Q_n^i \approx \frac{\lambda_1^i \cdots \lambda_{n-1} \lambda_n}{\lambda_1 \cdots \lambda_{n-1}} = \lambda_n$$

in the sense that the ratio of Q_n/λ_n is bounded from above and below by positive constants depending only on the dimension n . It follows from (2.17) that

$$\lambda_n(q, t) \geq c > 0, \quad \forall t \in [t_0 - \tau, t_0 + \tau], \quad \text{dist}_{M^n}(p, q) < \epsilon$$

where $c > 0$ depends only on d_p , Σ_0 and the dimension n . Since, λ_n is the smallest of the principal curvatures of Σ_t , our result readily follows. □

We conclude our discussion with the open questions: (i) Does the surface Σ_t become strictly convex before it becomes extinct? Or in the case that (i) is not valid: (ii) Does the surface Σ_t always shrink to a point?

3. PART II: THE EVOLUTION OF A SURFACE WITH FLAT SIDES

In this section we will study the evolution of a convex surface with flat sides under the $(*_k)$ flow for $2 \leq k \leq n$. Our goal is to give the proof of Theorem 1.5. Let us briefly outline the steps of its proof. The $(*_k)$ flow can be seen as a free boundary problem arising from the degeneracy near the flat side of the fully nonlinear parabolic PDE which describes the flow. Via a coordinate change we will show that solving this free boundary problem is equivalent to solving an *initial value problem* of the form

$$(IVP) \quad \begin{cases} Mw = 0 & \text{on } \mathcal{D} \times [0, \tau] \\ w = w_0 & \text{at } t = 0 \end{cases}$$

on the unit ball in \mathbb{R}^n

$$\mathcal{D} = \{x := (x_1, \dots, x_n) \in \mathbb{R}^n; |x| \leq 1\}.$$

The operator M , defined as

$$Mw = w_t - F(t, x_1, \dots, x_n, w, Dw, D^2w)$$

is a fully non-linear operator which becomes degenerate at $\partial\mathcal{D}$, the boundary of \mathcal{D} . We will apply the inverse function theorem between appropriately defined Banach spaces to show that this problem admits a solution on $0 \leq t \leq \tau$, for some $\tau > 0$.

The linearization of the operator M at a point \bar{w} close to the initial data w_0 can be modeled (after straightening the boundary near $z := x_1 = 0$) on the degenerate equation

$$(3.1) \quad f_t = z^2 \tilde{a}_{11} f_{11} + 2z \tilde{a}_{1i} f_{1i} + \tilde{a}_{ij} f_{ij} + z \tilde{b}_1 f_1 + \tilde{b}_i f_i + \tilde{c} f, \quad i, j \neq 1$$

on the half space $z > 0$ with no extra conditions on f along the boundary $z = 0$. The diffusion in the above equation is governed by the *singular* Riemannian metric

$$ds^2 = d\bar{s}^2 + |dt|$$

where

$$(3.2) \quad d\bar{s}^2 = \frac{dz^2}{z^2} + dx_2^2 + \cdots + dx_n^2.$$

We notice that the distance (with respect to the singular metric \bar{s}) of an interior point ($z > 0$) from the boundary ($z = 0$) is *infinite*. This distinguishes our problem from other, previously studied, degenerate free-boundary problems such as the degenerate Gauss curvature flow [6], [7] and the porous medium equation [5].

The results in this part are generalizations, in dimensions $n \geq 3$, of the results in [4] for the harmonic mean curvature flow in dimension $n = 2$. Their proofs are similar to the corresponding proofs in [4]. We will only give the main steps, referring the reader to [4] for the details.

3.1. Local Change of Coordinates. In Section 3.4 we will give the global change of coordinate which transforms our free-boundary problem $(*_k)$ to a degenerate problem of the form (IVP) on a domain with fixed boundary. Since the computations there are quite involved, to motivate our discussion we will present here a local change of coordinates near the interface which fixes the free boundary and we will give the definitions of the appropriate Banach spaces where the existence of solutions will be shown.

We will assume throughout this section that the surface Σ_0 belongs to the class \mathfrak{S} , as defined in the introduction. Let Σ_t be a solution to the $(*_k)$ flow on $[0, \tau)$, for some $\tau > 0$ such that $\Sigma_t = \Sigma_t^1 \cup \Sigma_t^2$, with Σ_t^1 flat and Σ_t^2 strictly convex. Let $P_0(x_1, \dots, x_n, 0)$ be a point on the interface Γ_{t_0} , for $t_0 > 0$ sufficiently small. Then, the strictly convex part of surface Σ_t^2 , $t < t_0$ can be expressed locally around P_0 as the graph of a function $z = u(x_1, \dots, x_n, t)$. Let $g = \sqrt{u}$ be the pressure function. Assuming that g is of class C^2 up to the interface and satisfies condition (\star) , then we solve locally around the point P_0 the equation $z = u(x_1, \dots, x_n, t)$ with

respect to x_1 . This yields to the map $x_1 = f(z, x_2, \dots, x_n, t)$. The condition (\star) on g expressed in terms of f gives the following *non-degeneracy condition* $(\star\star)$ in a small neighborhood of $z = 0$:

$$(\star\star) \quad \begin{pmatrix} -z^{\frac{3}{2}} f_{11} & z^{\frac{3}{4}} f_{12} & & z^{\frac{3}{4}} f_{12} \\ & z^{\frac{3}{4}} f_{12} & & \\ & & -F_{ij} & \\ & z^{\frac{3}{4}} f_{1n} & & \end{pmatrix} \geq \bar{\lambda} I$$

in the sense that the eigenvalues of the above matrix are bounded from below by a number $\bar{\lambda} > 0$. $\{F_{ij}\}$ is the Hessian matrix for the function f with respect to the tangential directions.

When Σ_t evolves by the $(*_k)$ flow, then the function f evolves by a fully-nonlinear evolution equation of the form

$$(3.3) \quad f_t = -\frac{S_k^n([b_{ij}])}{S_{k-1}^n([b_{ij}])}$$

where b_{ij} can be expressed in terms of f and its first and second derivatives. The details are given in Section 3.5. Equation (3.3) becomes degenerate near the interface $z = 0$. Its linearization near the interface $z = 0$ is of the form (3.1). Our goal is to construct a smooth solution to this equation by using the inverse function theorem between appropriately defined Banach spaces which are scaled according to the singular metric (3.2). The main step is to show existence in an appropriately weighted $C_{w,\bar{s}}^{2+\alpha}$ space, with respect to the metric \bar{s} . The definition of these spaces will be given in the next section. Once a $C_{w,\bar{s}}^{2+\alpha}$ solution is established, one can prove the existence of a C^∞ solution by repeated differentiation. Since this is similar to the results in [4] we will omit its proof.

3.2. The Hölder spaces with respect to the singular metric. We will define in this section the Banach spaces on which solutions of degenerate equations of the form (3.1) are naturally defined.

We will denote for the next of this section by \bar{x} points $\bar{x} := (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ and we will consider points $(z, \bar{x}) \in \mathbb{R}^n$. Let \mathcal{A} be a compact subset of the half space $\{(z, \bar{x}) \in \mathbb{R}^n : z \geq 0\}$ such that $0 \in \mathcal{A}$. We define:

$$\begin{aligned} \mathcal{A}^\circ &:= \{ \bar{x} \in \mathbb{R}^{n-1} : (0, \bar{x}) \in \mathcal{A} \} \\ Q_\tau &:= \mathcal{A} \times [0, \tau], \quad \tau > 0 \\ Q_\tau^\circ &:= \mathcal{A}^\circ \times [0, \tau], \quad \tau > 0. \end{aligned}$$

We define the *hyperbolic distance* $\bar{s}(P_1, P_2)$ between two points $P_1 = (z_1, \bar{x}_1)$ and $P_2 = (z_2, \bar{x}_2)$ in \mathcal{A} with $z_1 > 0, z_2 > 0$ to be:

$$\bar{s}(P_1, P_2) := \sqrt{|\ln z_1 - \ln z_2|^2 + |\bar{x}_1 - \bar{x}_2|^2}, \quad \text{if } 0 < z_1, z_2 \leq 1$$

otherwise it is defined to be equivalent to the standard euclidean metric.

We define the *parabolic hyperbolic distance* between two points $\tilde{P}_1 = (z_1, \bar{x}_1, t_1)$ and $\tilde{P}_2 = (z_2, \bar{x}_2, t_2)$ with $z_1 > 0, z_2 > 0$ to be:

$$s(\tilde{P}_1, \tilde{P}_2) := \bar{s}(P_1, P_2) + \sqrt{|t_1 - t_2|}$$

where $P_1 = (z_1, \bar{x}_1), P_2 = (z_2, \bar{x}_2)$.

We will next define Hölder function spaces with respect to the metric s .

Given a function f on \mathcal{A} we define:

$$f^\circ(\bar{x}) := f(0, \bar{x}), \quad \tilde{f}(z, \bar{x}) := \frac{1}{\sqrt{z}} (f(z, \bar{x}) - f^\circ(\bar{x}))$$

Analogously, given a function f on Q_τ we define:

$$f^\circ(\bar{x}, t) := f(0, \bar{x}, t), \quad \tilde{f}(z, \bar{x}, t) := \frac{1}{\sqrt{z}} (f(z, \bar{x}, t) - f^\circ(\bar{x}, t))$$

Let $0 < \alpha \leq 1$. We will define the weighted Hölder space $C_{w, \bar{s}}^\alpha(\mathcal{A})$ in terms of the above distance. We start defining the Hölder semi-norm:

$$\|f\|_{H_{\bar{s}}^\alpha(\mathcal{A})} := \sup_{P_1 \neq P_2 \in \bar{\mathcal{A}}} \frac{|f(P_1) - f(P_2)|}{\bar{s}[P_1, P_2]^\alpha}$$

and the norm

$$\|f\|_{C_{\bar{s}}^\alpha(\mathcal{A})} := \|f\|_{C^0(\mathcal{A})} + \|f\|_{H_{\bar{s}}^\alpha(\mathcal{A})}$$

where $\|f\|_{C^0(\mathcal{A})} := \sup_{P \in \mathcal{A}} |f(P)|$.

Moreover, we define:

$$\|f\|_{C_w^\alpha(\mathcal{A})} := \|f^\circ\|_{C^0(\mathcal{A}^\circ)} + \|\tilde{f}\|_{C^0(\mathcal{A})}.$$

Definition 3.1 (The space $C_{w, \bar{s}}^\alpha(\mathcal{A})$). *A function f belongs to $C_{w, \bar{s}}^\alpha(\mathcal{A})$ iff $f^\circ \in C^\alpha(\mathcal{A}^\circ)$ and $\tilde{f} \in C_{\bar{s}}^\alpha(\mathcal{A})$. The norm of f in the space $C_{w, \bar{s}}^\alpha(\mathcal{A})$ is defined as:*

$$\|f\|_{C_{w, \bar{s}}^\alpha(\mathcal{A})} := \|f^\circ\|_{C^\alpha(\mathcal{A}^\circ)} + \|\tilde{f}\|_{C_{\bar{s}}^\alpha(\mathcal{A})}.$$

Remark 3.2. *We observe that $f(z, \bar{x}) \in C_{\bar{s}}^\alpha(\mathcal{A})$ iff the function $\bar{f}(\xi, \bar{x}) := f(e^\xi, \bar{x})$ belongs to the space $C^\alpha(\bar{\mathcal{A}})$ (this is the Hölder space with respect to the standard metric) where $\bar{\mathcal{A}} := \{(\xi, \bar{x}) : (e^\xi, \bar{x}) \in \mathcal{A}\}$.*

We will next define weighted Hölder spaces of second order derivatives with respect to our metric \bar{s} .

Definition 3.3 (The space $C_w^2(\mathcal{A})$). *We say that a continuous function f on \mathcal{A} belongs to $C_w^2(\mathcal{A})$ if $f^\circ \in C^2(\mathcal{A}^\circ)$ and f has continuous derivatives*

$$f_z, f_i, f_{zz}, f_{zi}, f_{ij}, \quad i, j = 2, \dots, n$$

in the interior of \mathcal{A} such that

$$\frac{1}{\sqrt{z}}(f - f^\circ), \sqrt{z}f_z, \frac{1}{\sqrt{z}}(f_i - f_i^\circ), z^{\frac{3}{2}}f_{zz}, \sqrt{z}f_{zi}, \frac{1}{\sqrt{z}}(f_{ij} - f_{ij}^\circ)$$

extend continuously (with respect to the standard euclidean metric) up to the boundary $z = 0$, for all $i, j = 2, \dots, n$. The norm of f in the space $C_w^2(\mathcal{A})$ is defined as follows:

$$\|f\|_{C_w^2(\mathcal{A})} := \left\| \sum_{m=0}^2 D_{\bar{x}}^m f^\circ \right\|_{C^0(\mathcal{A}^\circ)} + \sum_{m+n=0}^2 \|z^m D_z^m D_{\bar{x}}^n \tilde{f}\|_{C^0(\mathcal{A})}.$$

Definition 3.4 (The space $C_{w,\bar{s}}^{2+\alpha}(\mathcal{A})$). *Given $f \in C_w^2(\mathcal{A})$, we say that f belongs to $C_{w,\bar{s}}^{2+\alpha}(\mathcal{A})$ if*

$$f^\circ \in C^{2+\alpha}(\mathcal{A}^\circ) \quad \text{and} \quad z f_z, f_i, z^2 f_{zz}, z f_{zi}, f_{ij}, \quad i, j = 2, \dots, n$$

extend continuously up to the boundary and the extensions are Hölder continuous on \mathcal{A} of class $C_{w,\bar{s}}^\alpha(\mathcal{A})$. The norm of f in the space $C_{w,\bar{s}}^{2+\alpha}(\mathcal{A})$ is defined as:

$$\|f\|_{C_{w,\bar{s}}^{2+\alpha}(\mathcal{A})} := \|f^\circ\|_{C^{2+\alpha}(\mathcal{A}^\circ)} + \sum_{m+n=0}^2 \|z^m D_z^m D_{\bar{x}}^n f\|_{C_{w,\bar{s}}^\alpha(\mathcal{A})}.$$

Let $\tau > 0$. Similarly as in [4], the definitions above can be naturally extended on a space-time domain $Q_\tau = \mathcal{A} \times [0, \tau]$ by using the parabolic distance $ds^2 = d\bar{s}^2 + |dt|$. We call the resulting spaces $C_{w,s}^\alpha(Q_\tau)$, $C_{w,s}^2(Q_\tau)$ and $C_{w,s}^{2+\alpha}(Q_\tau)$ respectively.

3.3. The Degenerate Equation on the disc. We will show in Section 3.4 that the initial value problem $(*_k)$ can be transformed, via a global coordinate change, to an initial value problem of the form (IVP). Its linearization at a point \bar{w} close to the initial date w_0 is a degenerate equation of the form

$$(3.4) \quad Lw := w_t - (a^{ij}w_{ij} + b^i w_i + cw) = 0$$

on the cylinder $\mathcal{D} \times [0, \tau)$, $\tau > 0$, where \mathcal{D} denotes the unit ball in \mathbb{R}^n . The matrix $\{a^{ij}\}$ is symmetric and under appropriate an change of coordinates near $\partial\mathcal{D}$

which straightens the boundary, equation (3.4) is transformed into the degenerate equation of the form (3.1).

We define the distance function s in \mathcal{D} as follows: in the interior of \mathcal{D} , s is equivalent to the standard euclidean distance, while around any boundary point $P \in \partial\mathcal{D}$, s is defined as the pull back of the distance function induced by the metric $d\bar{s}^2$ defined in (3.2) on the half space $\mathcal{S}_0 = \{(z, x_2, \dots, x_n) : z \geq 0\}$, via a map $\varphi : \mathcal{S}_0 \cap \mathcal{D} \rightarrow \mathcal{D}$ that flattens the boundary of the ball \mathcal{D} near P . We denote by ds^2 the associated parabolic distance.

We can now define the spaces $C_{w,\bar{s}}^\alpha(\mathcal{D})$ and $C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})$: for a fixed and small number δ in $0 < \delta < 1$, we write

$$\mathcal{D} = \mathcal{D}_{1-\delta/2} \cup \left(\bigcup_l (\mathcal{D}_\delta(P_l) \cap \mathcal{D}) \right)$$

for finite many points $P_l \in \partial\mathcal{D}$, $l \in I$, with $\mathcal{D}_{1-\delta/2}$ denoting the ball centered at the origin of radius $1 - \delta/2$ and $\mathcal{D}_\delta(P_l)$ denoting the ball of radius δ centered at P_l .

We denote by \mathcal{D}_+ the half disk $\mathcal{D}_+ = \{(z, \bar{x}) \in \mathcal{D} : z \geq 0\}$. We can choose charts $\Upsilon_l : \mathcal{D}_+ \rightarrow \mathcal{D}_\delta(P_l) \cap \mathcal{D}$ which flatten the boundary of \mathcal{D} and such that $\Upsilon_l(0) = P_l$, $l \in I$. Let $\{\psi, \psi_l\}$ be a partition of unity subordinated to the cover $\{\mathcal{D}_{1-\delta/2}, (\mathcal{D}_\delta(P_l) \cap \mathcal{D})\}$ of \mathcal{D} , with $l \in I$.

Definition 3.5 (The spaces $C_{w,\bar{s}}^\alpha(\mathcal{D})$ and $C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})$). *We define $C_{w,\bar{s}}^\alpha(\mathcal{D})$ to be the space of all functions w on \mathcal{D} such that $w \in C^\alpha(\mathcal{D}_{1-\delta/2})$ and $w \circ \Upsilon_l \in C_{w,\bar{s}}^\alpha(\mathcal{D}_+)$ for all $l \in I$. Also, we define $C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})$ to be the space of all functions w on \mathcal{D} such that $w \in C^{2+\alpha}(\mathcal{D}_{1-\delta/2})$ and $w \circ \psi_l \in C_{w,\bar{s}}^{2+\alpha}(\mathcal{D}_+)$ for all $l \in I$.*

In the above definition C^α and $C^{2+\alpha}$ denote the regular Hölder Spaces, while $C_{w,\bar{s}}^\alpha(\mathcal{D}_+)$ and $C_{w,\bar{s}}^{2+\alpha}(\mathcal{D}_+)$ denote the Hölder Spaces defined in Section 3.2. One can show that both spaces $C_{w,\bar{s}}^\alpha(\mathcal{D})$ and $C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})$ are Banach spaces under the norms

$$\|w\|_{C_{w,\bar{s}}^\alpha(\mathcal{D})} = \|\psi w\|_{C^\alpha(\mathcal{D}_{1-\delta/2})} + \sum_l \|\psi_l (w \circ \Upsilon_l)\|_{C_{w,\bar{s}}^\alpha(\mathcal{D}_+)}$$

and

$$\|w\|_{C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})} = \|\psi w\|_{C^{2+\alpha}(\mathcal{D}_{1-\delta/2})} + \sum_l \|\psi_l (w \circ \Upsilon_l)\|_{C_{w,\bar{s}}^{2+\alpha}(\mathcal{D}_+)}.$$

The above definitions can be extended in a straight forward manner to the parabolic spaces $C_{w,s}^\alpha(Q_\tau)$ and $C_{w,s}^{2+\alpha}(Q_\tau)$, where Q_τ is the cylinder $Q = \mathcal{D} \times [0, \tau]$, for some $\tau > 0$.

Before we state the main result in this section, we will give the assumptions on the coefficients of the equation (3.4) on the cylinder $Q = \mathcal{D} \times [0, T]$: We first assume that for any δ in $0 < \delta < 1$, the coefficients $\{a^{ij}\}$, $\{b^i\}$ and c belong to the Hölder class $C^\alpha(\mathcal{D}_{1-\delta/2} \times [0, T])$, which means that the coefficients are of the class C^α in the interior of \mathcal{D} . In addition we assume that the metric $\{a_{ij}\}$ is strictly elliptic in $\mathcal{D}_{1-\delta/2}$. For a number δ in $0 < \delta < 1$, let $\Upsilon_l : \mathcal{D}_+ \rightarrow \mathcal{D}_\delta(P_l) \cap \mathcal{D}$ be the collection of charts which flatten the boundary of \mathcal{D} , considered above. We assume that there exists a number δ so that for every $l \in I$, the coordinate change introduced by each of the Υ_l transforms the operator $L[w]$ defined in (3.4) on $\mathcal{D}_\delta(P_l) \cap \mathcal{D}$, into an operator \tilde{L}_l on \mathcal{D}_+ of the form (3.1) with the coefficients \tilde{a}_{ij} , \tilde{b}_i , \tilde{c} belonging to the class and with $\{\tilde{a}_{ij}\}$ strictly elliptic.

Theorem 3.6. *Assume that the operator L satisfies all the above conditions on the cylinder $Q_\tau = \mathcal{D} \times [0, \tau]$. Then, given any function $w_0 \in C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})$ and any function $g \in C_{w,s}^\alpha(Q)$ there exists a unique solution $w \in C_{w,s}^{2+\alpha}(Q_\tau)$ of the initial value problem*

$$\begin{cases} Lw = g & \text{in } Q \\ w(\cdot, 0) = w_0 & \text{on } \mathcal{D} \end{cases}$$

satisfying

$$(3.5) \quad \|w\|_{C_{w,s}^{2+\alpha}(Q)} \leq C(\tau) \left(\|w_0\|_{C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})} + \|g\|_{C_{w,s}^\alpha(Q)} \right).$$

The constant $C(\tau)$ depends only on the numbers α , τ , the ellipticity constant of $\{\tilde{a}_{ij}\}$ and the Hölder norms of the coefficients.

Proof. The proof follows the arguments of the proof of Theorem 4 in [4]. \square

Finally, the following existence result follows from Theorem 3.6 and the Inverse Function Theorem between Banach spaces.

Theorem 3.7. *Let w_0 be a function in $C_{w,\bar{s}}^{2+\alpha}(\mathcal{D})$. Assume that the linearization $DM(\bar{w})$ of the fully-nonlinear operator*

$$(3.6) \quad Mw = w_t - F(t, u, v, w, Dw, D^2w)$$

defined on the cylinder $Q = \mathcal{D} \times [0, T]$, satisfies the hypotheses of Theorem 3.6 at all points $\bar{w} \in C_{w,s}^{2+\alpha}(Q)$, with $\|\bar{w} - w_0\|_{C_{w,s}^{2+\alpha}(Q)} \leq \mu$, for some $\mu > 0$. Then, there exists a number τ_0 in $0 < \tau_0 \leq T$ depending on α , μ and the ellipticity of $\{\tilde{a}_{ij}\}$, for

which the initial value problem

$$(3.7) \quad \begin{cases} Mw = 0 & \text{in } \mathcal{D} \times [0, \tau_0] \\ w(\cdot, 0) = w_0 & \text{on } \mathcal{D} \end{cases}$$

admits a solution w in the space $C_{w,s}^{2+\alpha}(\mathcal{D} \times [0, \tau_0])$. Moreover,

$$\|w\|_{C_{w,s}^{2+\alpha}(\mathcal{D} \times [0, \tau_0])} \leq C \|w_0\|_{C_{w,s}^{2+\alpha}(\mathcal{D})}$$

for some positive constant C which depends only α , μ and the ellipticity of $\{\tilde{a}_{ij}\}$.

Proof. The proof follows the arguments of the proof of Theorem 5 in [4]. \square

3.4. Global change of coordinates and existence in $C_{w,s}^{2+\alpha}$. In this section we introduce a global change of coordinates which transforms the $(*_k)$ flow for a surface Σ_0 with flat sides into a fully-nonlinear degenerate parabolic PDE on the unit ball \mathcal{D} . We will only give a brief outline, as all the results are a straightforward generalization of the 2-dimensional case treated in [4].

We recall that our initial surface Σ_0 is of the form $\Sigma_0 = \Sigma_0^1 \cup \Sigma_0^2$ with Σ_0^1 flat and Σ_0^2 strictly convex. Let \mathcal{S}_0 be a smooth surface with boundary close to Σ_0^2 and such that $\partial\mathcal{S}_0 \subset \{x_{n+1} = 0\}$. Let $S_0 : \mathcal{D} \rightarrow \mathbb{R}^{n+1}$, indicate a parameterization of \mathcal{S}_0 on the unit disk \mathcal{D} , namely if $U = (u_1, \dots, u_n)$ and $X = (x_1, \dots, x_n)$ we have

$$S_0(U) = (X, z) \in \mathbb{R}^{n+1} \quad \text{which maps } \partial\mathcal{D} \text{ onto } \partial\mathcal{S}_0.$$

Let $\eta > 0$ be sufficiently small. Let $T = (T_1, T_2, \dots, T_{n+1})$ be a smooth vector field transverse to \mathcal{S}_0 which is parallel to the $x_{n+1} = 0$ plane in a small neighborhood near $\partial\mathcal{S}_0$. Following [4], we define the global change of coordinates $\Phi : \mathcal{D} \times [-\eta, \eta] \rightarrow \mathbb{R}^{n+1}$ by

$$(3.8) \quad \begin{pmatrix} X \\ z \end{pmatrix} = \Phi \begin{pmatrix} U \\ w \end{pmatrix} = S(U) + wT(U).$$

Theorem 3.8. *Assume that the initial surface Σ_0 satisfies the assumptions of Theorem 1.5. Then, there exists a time $\tau > 0$ for which the solution Σ_t of the $(*_k)$ flow is converted, via the coordinate change (3.8), to a solution $w(\cdot, t)$ of the IVP (3.7) on $Q_\tau = \mathcal{D} \times [0, \tau]$ with $w_0 \in C_{w,s}^{2+\alpha}(\mathcal{D})$ and the operator M satisfying all the hypotheses of Theorem 3.7.*

Proof. The proof of this result follows along the lines of the proof of Theorem 6 in [4]. It is clear that the coordinate change (3.8) transforms the free-boundary problem $(*_k)$ into a problem of the form (3.7) where the fully-nonlinear operator M is strictly elliptic away from the lateral boundary $\partial\mathcal{D} \times [0, \tau]$. The main difficulty here is to show that the linearization of M at a point \bar{w} close to the initial data w_0 is a degenerate operator of the form (3.4) which near the boundary lateral boundary $\partial\mathcal{D} \times [0, \tau]$ satisfies the assumptions of Theorem 3.6. In the two-dimensional case, this is done in detail in [4] (Theorem 6). In Section 3.6 we will show that the linearization of equation (3.3), which is obtained from $(*_k)$ via a local change of coordinates that fixes the free-boundary, is of the desired form. The computations for the global change of coordinates are more involved by similar.

□

As an immediate consequence of Theorems 3.6 and 3.7 we obtain the following existence result for $(*_k)$.

Theorem 3.9. *Assume that the initial surface Σ_0 satisfies the assumptions of Theorem 1.5. Then, there exists a time $\tau > 0$ for which there exists a solution Σ_t of $(*_k)$ on $[0, \tau]$ which is of the class $C_{w,s}^{2+\alpha}$.*

Once a $C_{w,s}^{2+\alpha}$ solution of $(*_k)$ is established, one can argue similarly as in [4] that the pressure $g(\cdot, t) = \sqrt{u(\cdot, t)}$ of the solution Σ_t (as defined in the introduction) is C^∞ smooth up to the interface. This concludes the regularity part of Theorem 1.5, which we state next.

Theorem 3.10. *Assume that the initial surface Σ_0 satisfies the assumptions of Theorem 1.5. Let Σ_t be the unique viscosity solution of $(*_k)$ with initial data Σ_0 (its existence follows from Theorem 1.2). Then, there exists a time $\tau > 0$ such that the pressure function $g(\cdot, t) = \sqrt{u(\cdot, t)}$ is smooth up to the interface $x_{n+1} = 0$ and satisfies condition (\star) .*

3.5. Proof of Theorem 1.5. The proof of Theorem 1.5 readily follows from Theorem 3.10 and the following proposition that refers to the evolution of the boundary of the flat side.

Proposition 3.11. *Under the assumption that Σ_t belongs to the class \mathfrak{S} (as defined in Definition 1.3) and that the non-degeneracy condition (\star) holds on $0 \leq t \leq \tau$, then the boundary of the flat side Γ_t is an $(n-1)$ -dimensional surface which evolves by the $(*_k)$ flow.*

Proof. Let $P_0 \in \Gamma_t$ be a point on the boundary of the flat side. The strictly convex part of the surface Σ_t^2 can be expressed locally around P_0 as the graph of $z = u(x_1, \dots, x_n, t)$, where we may assume that the coordinates are chosen so that x_2, \dots, x_n are the tangential directions to the flat side at P_0 . We consider, as before, the pressure function $g = \sqrt{u}$ and we solve the equation $z = u(x_1, \dots, x_n, t)$ with respect to x_1 . Then $x_1 = f(z, x_2, \dots, x_n, t)$.

We observe that because of the non-degeneracy condition (\star) , we have

$$(3.9) \quad f_{x_i}(P_0) = 0, \quad \forall i \geq 2.$$

Indeed, along the tangential directions x_i ($i \geq 2$) to Γ_t at P_0 , we have

$$f_{x_i} = -\frac{u_{x_i}}{u_{x_1}} = -\frac{g_{x_i}}{g_{x_1}} = 0, \quad \text{at } P_0$$

because $g_{x_i}(P_0) = 0$, for $i \geq 2$ while $g_{x_1}(P_0) > 0$ due to (\star) .

The assumptions that $\Sigma_t \in \mathfrak{S}$ and that the non-degeneracy condition holds, imply in particular that $f \in C^{2+\alpha, \frac{1}{2}}$ near P_0 . Hence, by Definition 3.3 the functions $\frac{1}{\sqrt{z}} f_i$ extend continuously up to the boundary $z = 0$, when $i \geq 2$, which means that

$$(3.10) \quad |f_{x_i}| \leq C\sqrt{z}, \quad \text{as } z \rightarrow P_0, \quad \forall i \geq 2.$$

Our surface is locally expressed as a graph $z = u(x_1, \dots, x_n, t)$ and its principal curvatures are the eigenvalues of the symmetric matrix $[a_{ij}]$, where

$$(3.11) \quad a_{ij} = \frac{1}{v} \left(D_{ij}u - \frac{D_i u D_l u D_{jl} u}{v(1+v)} - \frac{D_j u D_l u D_{il} u}{v(1+v)} + \frac{D_i u D_j u D_k u D_l u D_{kl} u}{v^2(1+v)^2} \right)$$

and $v = \sqrt{1 + |Du|^2}$. The equation $(*_k)$ can be expressed as

$$u_t = \frac{S_k^n([c_{ij}])}{S_{k-1}^n([c_{ij}])}, \quad \text{with } c_{ij} = a_{ij} v.$$

Moreover, since $f_t = -u_t/f_z$, the equation $(*_k)$ can be written in terms of f as

$$(3.12) \quad f_t = -\frac{S_k^n([b_{ij}])}{S_{k-1}^n([b_{ij}])}, \quad \text{with } b_{ij} = c_{ij} f_z$$

where b_{ij} can be expressed in terms of f and its first and second derivatives.

To express the evolution equation in terms of f we use the identities

$$u_{x_1} = \frac{1}{f_z}, \quad u_{x_i} = -\frac{f_{x_i}}{f_z}, \quad u_t = -\frac{f_t}{f_z}$$

and

$$u_{x_1 x_1} = -\frac{f_{zz}}{f_z^3}, \quad u_{x_1 x_i} = -\frac{1}{f_z} \left(-\frac{f_{x_i}}{f_z^2} f_{zz} + \frac{1}{f_z} f_{z x_i} \right)$$

and

$$u_{x_i x_j} = -\frac{1}{f_z} \left(\frac{f_{x_i} f_{x_j}}{f_z^2} f_{zz} - \frac{f_{x_i}}{f_z} f_{zx_i} - \frac{f_{x_j}}{f_z} f_{zx_j} + f_{x_i x_j} \right)$$

where $i, j > 1$.

Keeping in mind Definition 3.3 and that we are interested in a behaviour of $[b_{ij}]$ around $z = 0$, using (3.11) and the above formulas we can compute that, as $z \rightarrow 0$, we have

$$b_{11} = -\frac{f_{zz}}{f_z^2} + o(1), \quad b_{1i} = b_{i1} = -\frac{f_{zx_i}}{f_z} + \frac{f_{x_i} f_{zz}}{f_z^2} + o(1)$$

and

$$b_{ij} = -f_{x_i x_j} + \frac{f_{x_i} f_{zx_j}}{f_z} + \frac{f_{x_j} f_{zx_i}}{f_z} - \frac{f_{x_i} f_{x_j} f_{zz}}{f_z^2} + o(1)$$

for $i, j \geq 2$.

To simplify the notation we set from now on $\bar{x} = (x_2, \dots, x_n)$. By Definition 3.3 we can see that the behaviour of $[b_{ij}]$, as $z \rightarrow 0$, is

$$b_{11} = \frac{c_1(z, \bar{x}, t)}{\sqrt{z}}, \quad b_{1i} = b_{i1} = c_i(z, \bar{x}, t), \quad b_{ij} = -f_{ij} + c_{ij}(z, \bar{x}, t), \quad i, j \geq 2$$

with

$$|c_i(z, \bar{x}, t)| \leq C \quad \text{and} \quad \lim_{z \rightarrow 0} c_{ij}(z, \bar{x}, t) = 0$$

and in addition, by the non-degeneracy condition $(\star\star)$,

$$0 < \delta \leq c_1(z, \bar{x}, t) \leq C.$$

We may assume that the coordinates $\bar{x} := (x_2, \dots, x_n)$ are chosen so that the matrix $[f_{ij}(P_0)]_{i,j \geq 2}$ is diagonal at our chosen boundary P_0 . Then, it follows from the above discussion that

$$(3.13) \quad b_{ii} = -f_{ii} + o(1), \quad (b_{ij})_{i \neq j} = o(1), \quad i, j \geq 2.$$

The non-degeneracy condition $(\star\star)$ and our regularity assumption on f imply that

$$(3.14) \quad 0 < \lambda \leq b_{ii} \leq C < \infty, \quad (b_{ij})_{i \neq j} = o(1), \quad i, j \geq 2.$$

It follows that the eigenvalues of $[b_{ij}]$ are computed as the roots of a polynomial in λ which is of the form

$$(3.15) \quad \det([b_{ij}] - \lambda I) = \left(\frac{c_1}{\sqrt{z}} \prod_{j=2}^n (b_{jj} - \lambda) - \sum_{i=2}^n c_i^2 \prod_{j \neq i, j \geq 2} (b_{jj} - \lambda) \right) (1 + o(1)).$$

Denote the eigenvalues of the matrix $[b_{ij}]$ by

$$\lambda_1(z, \bar{x}, t) \geq \dots \geq \lambda_n(z, \bar{x}, t).$$

Claim 3.12. *There are uniform constants $\mu > 0$ and $\nu < \infty$ so that*

$$(3.16) \quad \mu \leq \frac{\lambda_1(z, \bar{x}, t)}{\sqrt{z}}, \quad \lambda_i(z, \bar{x}, t) \leq \nu, \quad i \geq 2.$$

Proof. We will first show there is a uniform constant $\mu > 0$ so that

$$(3.17) \quad \lambda_i \geq \mu, \quad \text{for all } i \geq 1.$$

Assume, by contradiction, that there is a λ_k so that $\lambda_k(z, \bar{x}, t) \rightarrow 0$, as $z \rightarrow 0$. Since, $\det([b_{ij}] - \lambda_k I) = 0$, by applying (3.15) with $\lambda = \lambda_k$ and using (3.14) we obtain

$$(3.18) \quad \lim_{z \rightarrow 0} \left(\frac{c_1(z, \bar{x}, t)}{\sqrt{z}} \prod_{i=2}^n b_{ii}(z, \bar{x}, t) - \sum_{i=2}^n c_i(z, \bar{x}, t)^2 \prod_{j \neq i, j \geq 2} b_{jj}(z, \bar{x}, t) \right) = 0$$

which is impossible by (3.14) and the fact that all b_{ii} , c_i are bounded and $c_1 \geq \delta > 0$.

Also, note that

$$\prod_{i=1}^n \lambda_i = \det([b_{ij}]) = \left(\frac{c_1}{\sqrt{z}} \prod_{j>1} b_{jj} - \sum_{i>1} c_i^2 \prod_{i \neq j, j \geq 2} b_{jj} \right) (1 + o(1))$$

and

$$\sum_{i=1}^n \lambda_i = \frac{c_1}{\sqrt{z}} + b_{22} + \cdots + b_{nn} + o(1)$$

are both of the order $z^{-\frac{1}{2}}$ as $z \rightarrow 0$. Hence by (3.17) and (3.14) we conclude the bounds (3.16) finishing the proof of our claim. \square

Claim 3.13. *After a possible index re-arrangement we have*

$$\lim_{z \rightarrow 0} (f_{ii}(z, \bar{x}, t) - \lambda_i(z, \bar{x}, t)) = 0, \quad \forall i \geq 2.$$

Proof. Let λ_k , $k \geq 2$ be any eigenvalue. Then, using (3.15) we have

$$\lim_{z \rightarrow 0} \left(\frac{c_1}{\sqrt{z}} \prod_{j \geq 2} (b_{jj} - \lambda_k) - \sum_{i \geq 2} c_i^2 \prod_{j \neq i, j \geq 2} (b_{jj} - \lambda_k) \right) = 0$$

and since the second term is bounded, we conclude using also (3.14) and (3.13) that

$$(3.19) \quad \lim_{z \rightarrow 0} (f_{ii}(z, \bar{x}, t) - \lambda_k(z, \bar{x}, t)) = 0$$

for some $i \geq 2$.

It remains to show that, vice versa, for every $i \geq 2$, there exists an eigenvalue λ_k , $k \geq 2$, so that (3.19) holds. We will argue by contradiction. Assume that

there exists an $i \geq 2$ so that (3.19) fails to hold for all λ_k , $k \geq 2$. Without loss of generality, let's assume that $i = n$, so that

$$\lim_{z \rightarrow 0} (f_{nn}(z, \bar{x}, t) - \lambda_k(z, \bar{x}, t)) \neq 0, \quad \forall k \geq 2.$$

We will look at an eigenvector $V := (v_1, \dots, v_n)$ corresponding to any $\lambda \in \{\lambda_2, \dots, \lambda_n\}$. Using our previous computations on the behavior of the matrix $[b_{ij}]$, it follows that the coordinates of V satisfy the following system of equations

$$\begin{aligned} \frac{c_1}{\sqrt{z}} v_1 + c_2 v_2 + \dots + c_n v_n + o(1) &= \lambda v_1 \\ c_j v_1 + f_{jj} v_j + o(1) &= \lambda v_j, \quad \forall j \geq 2. \end{aligned}$$

Since λ and all the coefficients $c_i(z, \bar{x}, t)$ are bounded as $z \rightarrow 0$, the first equation implies the $\lim_{z \rightarrow 0} v_1(z, \bar{x}, t) = 0$. The last equation gives that

$$c_n v_1 = (\lambda - f_{nn}) v_n + o(1), \quad \text{as } z \rightarrow 0$$

and since the $\lim_{z \rightarrow 0} (f_{nn} - \lambda) \neq 0$ it implies that the $\lim_{z \rightarrow 0} v_n(z, \bar{x}, t) = 0$. We conclude that all the eigenvectors corresponding to $\lambda_2, \dots, \lambda_n$ are of the form

$$V^i = (o(1), v_2^i, \dots, v_{n-1}^i, o(1)), \quad \text{as } z \rightarrow 0.$$

We will argue that this is impossible because the V_i^i 's at the limit $z \rightarrow 0$ must span the tangent plane to the surface $z = f(0, \bar{x}, t)$, which is $(n-1)$ -dimensional.

To this end, we consider the slices $x_1 = f(z, \bar{x}, t)$, when z is fixed, but close to zero. Since x_2, \dots, x_n are the tangential directions to the flat side at P_0 and since the slices $x_1 = f(z, \bar{x}, t)$ converge nicely to the flat side $x_1 = f(0, \bar{x}, t)$ (from our regularity assumptions on f), it follows that x_2, \dots, x_n are almost tangential directions to the slice $x_1 = f(z, x_2, \dots, x_n, t)$, when z is close to zero. Therefore the eigenvectors V^2, \dots, V^n span an $(n-1)$ -dimensional plane that is almost tangent to the graph $x_1 = f(z, x_2, \dots, x_n, t)$. Because of the nice convergence of the slices to the flat side, those almost tangent planes converge to the tangent plane to the interface $x_1 = f(0, \bar{x}, t)$ at P_0 .

On the other hand, we observe that each V^i converges, as $z \rightarrow 0$, to a vector of the form

$$\bar{V}^i = (0, \bar{v}_2^i, \dots, \bar{v}_{n-1}^i, 0).$$

The span of $\langle V^2, \dots, V^n \rangle$ converges to a span of $\langle \bar{V}^2, \dots, \bar{V}^n \rangle$, which is at most $(n-2)$ -dimensional and therefore it is impossible to define the tangent plane to $x_1 = f(0, x_2, \dots, x_n, t)$. This finishes the proof of our claim. \square

Claim 3.14. *The principal curvatures of the interface $x_1 = f(0, x_2, \dots, x_n, t)$ are given by*

$$\bar{\lambda}_i = f_{ii}(0, x_2, \dots, x_n, t), \quad i \geq 2.$$

Proof. Since the interface is the graph of the function $x_1 = f(0, x_2, \dots, x_n, t)$, its principal curvatures can be computed by using formula (3.11). Since $\nabla f(P_0) = 0$, the principal curvatures of the interface are the eigenvalues of the matrix $[f_{ij}]$. By our choice of coordinates at P_0 this matrix is diagonal, which proves the claim. \square

We will now conclude the proof of Proposition 3.11. Denote by $\bar{\lambda}_i(\bar{x}, t)$ the principal curvatures of the interface $z = 0$. By Claims 3.13 and 3.14, we have

$$\bar{\lambda}_i(\bar{x}, t) = \lim_{z \rightarrow 0} \lambda_i(z, \bar{x}, t), \quad \forall i \geq 2.$$

Since $\lim_{z \rightarrow 0} \lambda_1(z, \bar{x}, t) = \infty$, by L'Hospital's rule, we obtain

$$\begin{aligned} f_t(0, \bar{x}, t) &= - \lim_{z \rightarrow 0} \frac{S_k^n([b_{ij}])}{S_{k-1}^n([b_{ij}])} = - \lim_{z \rightarrow 0} \frac{\frac{\partial}{\partial \lambda_1} S_k^n([b_{ij}])}{\frac{\partial}{\partial \lambda_1} S_{k-1}^n([b_{ij}])} \\ &= - \lim_{z \rightarrow 0} \frac{S_{k-1}^{n-1}(\lambda_2, \dots, \lambda_n)}{S_{k-2}^{n-1}(\lambda_2, \dots, \lambda_n)} = - \frac{S_{k-1}^{n-1}(\bar{\lambda}_2, \dots, \bar{\lambda}_n)}{S_{k-2}^{n-1}(\bar{\lambda}_2, \dots, \bar{\lambda}_n)} \end{aligned}$$

which shows that the interface Γ_t shrinks by the $(*_k)$ flow. \square

We finally remark that we have actually shown the following stronger result, where we relax the regularity assumptions on the initial surface.

Theorem 3.15. *Assume that the initial surface Σ_0 belongs to the class $C_s^{2+\alpha, \frac{1}{2}}$ and satisfies the non-degeneracy conditions $(\star\star)$. Then, there exists a $\tau > 0$ for which the $(*_k)$ flow with initial data the surface Σ_0 admits a solution Σ_t which is smooth up to the interface, for $0 < t \leq \tau$. In particular, the interface Γ_t is a smooth hypersurface for every $0 < t \leq \tau$ which moves by the $(*_k)$ flow.*

3.6. Appendix. In this appendix we will justify why the linearization of the equation (3.12) satisfied by $x_1 = f(z, x_2, \dots, x_n)$ is of the form (3.1).

Proposition 3.16. *The linearization of (3.12) around a point $\tilde{f} \in C_{w,s}^{2+\alpha}$ which satisfies the non-degeneracy condition $(\star\star)$ is of the form*

$$\tilde{f}_t = z^2 \tilde{a}_{11} \tilde{f}_{11} + 2z \tilde{a}_{1i} \tilde{f}_{1i} + \tilde{a}_{ij} \tilde{f}_{ij} + z \tilde{b}_1 \tilde{f}_1 + \tilde{b}_i \tilde{f}_i + \tilde{c} \tilde{f}, \quad i, j \neq 1,$$

where $[\tilde{a}_{ij}]$ is a positive definite matrix.

Proof. The proof of the proposition relies on a computation done with mathematica and we will just briefly outline its steps. Let the linearization of (3.12) around a point $f \in C_{w,s}^{2+\alpha}$ which satisfies the non-degeneracy condition $(\star\star)$, be

$$\tilde{f}_t = a_{11}\tilde{f}_{11} + 2a_{1i}\tilde{f}_{1i} + a_{ij}\tilde{f}_{ij} + b_1\tilde{f}_1 + b_i\tilde{f}_i + c\tilde{f}, \quad i, j \neq 1.$$

Notice that the linearized coefficients are given by

$$(3.20) \quad a_{ij} = \sum_{p=1}^n \frac{\partial Q_k}{\partial \lambda_p} \frac{\partial \lambda_p}{\partial f_{ij}}, \quad b_i = \sum_{p=1}^n \frac{\partial Q_k}{\partial \lambda_p} \frac{\partial \lambda_p}{\partial f_i}, \quad c = \sum_{p=1}^n \frac{\partial Q_k}{\partial \lambda_p} \frac{\partial \lambda_p}{\partial f}.$$

It has been computed in [9] that

$$\begin{aligned} \frac{\partial Q_k}{\partial \lambda_p} &= \frac{1}{S_{k-1}^n(\lambda)^2} (S_{k-1,i}^n(\lambda)^2 - S_{k,i}^n(\lambda)S_{k-2,i}^n(\lambda)) \\ &\geq \frac{n}{k(n-k+1)} \left(\frac{S_{k-1,i}^n(\lambda)}{S_{k-1}^n(\lambda)} \right)^2 \end{aligned}$$

where $S_{k,i}^n(\lambda)$ denotes the sum of all terms of $S_k^n(\lambda)$ not containing the factor λ_i . It follows by the Claim 3.12 and the above inequality that there are uniform constants $\tilde{C}_1, \tilde{C}_2 > 0$ so that

$$\frac{\tilde{C}_1}{\lambda_1^2} \leq \frac{\partial Q_k}{\partial \lambda_1} \leq \frac{\tilde{C}_2}{\lambda_1^2} \quad \text{and} \quad \tilde{C}_1 \leq \frac{\partial Q_k}{\partial \lambda_p} \leq \tilde{C}_2, \quad \forall p \geq 2.$$

Hence, by Claim 3.12, we have

$$(3.21) \quad C_1 z \leq \frac{\partial Q_k}{\partial \lambda_1} \leq C_2 z \quad \text{and} \quad C_1 \leq \frac{\partial Q_k}{\partial \lambda_p} \leq C_2, \quad \forall p \geq 2$$

for uniform $C_1, C_2 > 0$. Since $f \in C_{w,s}^{2+\alpha}$ and satisfies the non-degeneracy condition $(\star\star)$, it follows by Definition 3.3 that

$$(3.22) \quad \sqrt{z} f_z = a(z), \quad z^{\frac{3}{2}} f_{zz} = -v(z)$$

and

$$(3.23) \quad \frac{1}{\sqrt{z}} f_{x_i} = h_i(z), \quad \sqrt{z} f_{zx_i} = u_i(z), \quad f_{x_i x_i} = -C_i(z), \quad 2 \leq i \leq n$$

where $v(z), a(z), C_i(z), h_i(z), u_i(z)$ are continuous functions at $z = 0$ and

$$(3.24) \quad v(z), a(z), C_i(z) \geq \delta > 0.$$

Claim 3.17. *The coefficient a_{11} satisfies $a_{11} = z^2 \tilde{a}_{11} (1 + o(1))$, where $\tilde{a}_{11}(z)$ is a strictly positive and continuous function in a neighborhood of $z = 0$.*

Proof. By (3.21) we have

$$a_{11} = \left(z \frac{\partial \lambda_1}{\partial f_{11}} + \sum_{p \geq 2} C_i \frac{\partial \lambda_p}{\partial f_{11}} \right) (1 + o(1))$$

where $C_i(z) \geq C_i > 0$ in a neighborhood of $z = 0$. To compute $\frac{\partial \lambda_p}{\partial f_{1k}}$ we will use the fact that λ_p are the eigenvalues of $[b_{ij}]$, namely

$$(3.25) \quad \det(b_{ij} - \lambda_p I_{ij}) = 0$$

where the matrix $[b_{ij}]$ is defined in section 3.5. If we differentiate (3.25) with respect to f_{1k} we get

$$(3.26) \quad \frac{\partial \lambda_p}{\partial f_{1k}} = \frac{\sum_{i,j} M^{ij} \frac{\partial b_{ij}}{\partial f_{1k}}}{\sum_i M^{ii}}$$

where M^{ij} denotes the ij -th minor of the matrix $[b_{ij} - \lambda_p I_{ij}]$. A direct calculation yields to

$$\frac{\partial b_{11}}{\partial f_{11}} = -\frac{1}{f_z^2} + O(z^3), \quad \frac{\partial b_{1i}}{\partial f_{11}} = \frac{f_{x_i}}{f_z^2} + O(z^3), \quad i \geq 2$$

and

$$\frac{\partial b_{jj}}{\partial f_{11}} = -\frac{f_{x_j}^2}{f_z^2} + O(z^3), \quad j \geq 2, \quad \frac{\partial b_{ij}}{\partial f_{11}} = -\frac{f_{x_i} f_{x_j}}{f_z^2} + O(z^3), \quad i \neq j \geq 2.$$

Moreover, by (3.26) and (3.22)-(3.23) we have

$$(3.27) \quad \frac{\partial \lambda_1}{\partial f_{11}} = -\frac{z}{a^2(z)} (1 + o(1)).$$

We claim that for $j \geq 2$,

$$b_{jj} - \lambda_j = e_j(z) \sqrt{z}, \quad e_j(z) \geq 0 \quad \text{in a neighborhood of } z = 0.$$

To this end, we notice that as in (3.15) we have

$$(3.28) \quad \frac{c_1(z)}{\sqrt{z}} \prod_{j \geq 2} (b_{jj} - \lambda_p) = \sum_{i \geq 2} c_i^2 \prod_{j \neq i, j \geq 2} (b_{jj} - \lambda_p) (1 + o(1))$$

where $c_1(z) > 0$ in a neighborhood of $z = 0$ and $p \geq 2$. We may assume $b_{ii}(z) \neq b_{pp}(z)$, $i \neq p$, since otherwise we can divide (3.28) by $(b_{pp} - \lambda_p)^k$, if there are $k+1$ indices i_1, \dots, i_{k+1} so that $b_{i_s i_s} = b_{pp}$ for $1 \leq s \leq k+1$. We can rewrite (3.28) as

$$\frac{c_1(z)}{\sqrt{z}} - \sum_{2 \leq i \neq p} \frac{c_i^2}{b_{ii} - \lambda_p} = \frac{c_p^2}{b_{pp} - \lambda_p}.$$

Since all the terms in the sum on the left hand side are bounded and since the coefficient $c_1(z) \geq \delta > 0$, the left hand side is strictly positive in a neighborhood of

$z = 0$. This implies the bound $b_{pp} - \lambda_p \geq 0$ in a neighborhood of $z = 0$. By direct calculation we have

$$(3.29) \quad \frac{\partial \lambda_p}{\partial f_{11}} = -\frac{z^2}{a(z)^2 v(z)} (a(z)^2 e(z) + 2a(z)u_i(z)h_i(z) + 3h_i^2(z)v(z)) (1 + o(1)).$$

We claim $a(z)u_i(z)h_i(z) \geq 0$ in a neighborhood of $z = 0$. To this end, we recall that by (3.24), we have $a(z) \geq \delta$ and that by (3.23) we have

$$u_i(z)h_i(z) = \sqrt{z} f_{zx_i} \frac{f_{x_i}}{\sqrt{z}} = \frac{(f_{x_i}^2)_z}{2}.$$

Since $f_{x_i}^2(0, x_2, \dots, x_n, t) = 0$, $f_{x_i}^2 \geq 0$ and since u_i, h_i are continuous functions at $z = 0$, we conclude that $u_i(z)h_i(z) \geq 0$ in a neighborhood of $z = 0$.

Combining (3.20), (3.21), (3.27) and (3.29), we conclude that

$$a_{11} = \frac{z^2 [v(z) + \sum_{i \geq 2} (2a(z)^2 e_i(z) + 2a(z)u_i(z)h_i(z) + 3v(z)h_i^2(z))]}{a(z)^2 v(z)} (1 + o(1))$$

that is, $a_{11} = \tilde{a}_{11} z^2$, where

$$\tilde{a}_{11}(z) = \frac{(v(z) + \sum_{i \geq 2} (2a(z)^2 e_i(z) + 2a(z)u_i(z)h_i(z) + 3v(z)h_i^2(z)))}{a^2(z)v(z)} (1 + o(1))$$

and hence

$$\tilde{a}_{11}(z) \geq \frac{1}{2a^2(z)}.$$

□

Claim 3.18. *The coefficients a_{ii} , $i \geq 2$, are continuous and satisfy the lower bound $a_{ii}(z) \geq \delta > 0$ in a neighborhood of $z = 0$.*

Proof. To simplify the notation, let us assume that $i = 2$. Similarly as before, we have

$$(3.30) \quad a_{22} = \sum_{p \geq 1} \frac{\partial Q_k}{\partial \lambda_p} \frac{\partial \lambda_p}{\partial f_{22}}.$$

A direct calculation shows that

$$\frac{\partial b_{jj}}{\partial f_{22}} = -(1 + o(1)), \quad \text{for } j \geq 2, \quad \frac{\partial b_{ij}}{\partial f_{22}} = o(1) \quad \text{in all other cases.}$$

Similar analysis as before yields to

$$(3.31) \quad \frac{\partial \lambda_1}{\partial f_{22}} = O(z), \quad \frac{\partial \lambda_p}{\partial f_{22}} = -(1 + o(1)), \quad p \geq 2.$$

By (3.21), (3.30) and (3.31) we get

$$a_{22}(z) = z O(z) + \sum_{p \geq 2} \eta_p(z) (1 + o(1))$$

where $\eta_p(z) \geq C_1 > 0$. This immediately implies the claim. □

The behaviour of all other linearized coefficients is obtained similarly. This finishes the proof of Proposition 3.16. \square

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