Global regularity of solutions to systems of reaction-diffusion with Sub-Quadratic Growth in any dimension

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Abstract

This paper is devoted to the study of the regularity of solutions to some systems of reaction–diffusion equations, with reaction terms having a subquadratic growth. We show the global boundedness and regularity of solutions, without smallness assumptions, in any dimension N. The proof is based on blow-up techniques. The natural entropy of the system plays a crucial role in the analysis. It allows us to use of De Giorgi type methods introduced for elliptic regularity with rough coefficients. In spite these systems are entropy supercritical, it is possible to control the hypothetical blow-ups, in the critical scaling, via a very weak norm. Analogies with the Navier-Stokes equation are briefly discussed in the introduction.

Key words. Reaction-diffusion systems. Global regularity. Blow-up methods.

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1 Introduction

This paper is dedicated to the study of the global regularity in time of the solutions to a class of reaction-diffusion systems. Reaction-diffusion systems are used as models for a variety of problems, especially in chemistry and biology [6, 11, 13, 14, 21, 25]. The question of the existence of global

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solutions is particularly important and it has been widely studied [20, 22, 24, 27, 30]. In full generality, such system can have solutions which blow up in finite time [26] as it is well known when considering non linear heat equations [15, 34]. In this article we focus on systems which are entropy supercritical, that is, such that the preserved physical quantities, as the mass and the entropy, are not shrinking (not even preserved) via the universal scaling of the system. The global regularity for entropy (or energy) supercritical problems is an important question in several areas like nonlinear waves, Schroedinger, and the Navier-Stokes equations in dimension $N \geq 3$.

We present, here, a class of systems for which we are able to show global regularity despite their supercritical nature.

We consider systems of the form:

$$\begin{cases} \partial_t a - D\Delta a = Q(a), \qquad t > 0,\ x \in \mathbb{R}^N, \\ a(0,x) = (a_1^0(x), a_2^0(x), \cdots, a_P^0(x)), \qquad x \in \mathbb{R}^N, \qquad P \geq 1. \end{cases} \tag{1}$$

We assume that the matrix of diffusion is diagonal:

$$D = \operatorname{diag}(D_1, \cdots, D_P), \qquad D_i \in \mathbb{R},\ \text{for}\ 1 \leq i \leq P,$$

with

$$\underline{d} := \inf_{1\leq i\leq P}\{D_i\} > 0 \qquad \text{and} \qquad \bar{d} := \sup_{1\leq i\leq P}\{D_i\}.$$

In addition, we assume that the reaction term Q is regular and it satisfies the following four conditions: there exists $0 < \nu < 2$ and $\Lambda > 0$ such that

$$Q_i(a) \geq 0, \quad \text{if}\ \ a_i \leq 0, \quad 1 \leq i \leq P, \tag{2}$$

$$|\nabla Q_i(a)| \leq \Lambda |a|^{\nu-1}, \qquad a \in \mathbb{R}_+^P, \quad 1 \leq i \leq P, \tag{3}$$

$$\sum_{i=1}^P Q_i(a) = 0, \qquad a \in \mathbb{R}_+^P, \tag{4}$$

$$\sum_{i=1}^P \ln a_i Q_i(a) \leq 0, \qquad a \in \mathbb{R}_+^P. \tag{5}$$

Hypothesis (2) ensures the nonnegativity of the a_i, hypothesis (3) is a restriction on the growth of Q, hypothesis (4) ensures the conservation of the total mass $\sum_{i=1}^P \int a_i\, dx$, and hypothesis (5) ensures the non increase of the entropy $\sum_i \int a_i \ln a_i\, dx$. Note that (3) is the only restrictive hypothesis. It requires that the system is at most subquadratic. The other hypothesis are physical and pretty standard in systems coming from irreversible chemistry phenomena.

Examples of systems verifying such hypothesis for $P=4$ are given by:

$$Q_i(a) = (-1)^i(\phi(a_1a_3) - \phi(a_2a_4)), \qquad a = (a_1, a_2, a_3, a_4) \in \mathbb{R}_+^4, \ \ 1 \le i \le 4,$$

where $\phi \in C^\infty(\mathbb{R}^+)$ is any nondecreasing function verifying

$$\begin{aligned} \phi(z) &= 0, \qquad \text{for} \ \ z < 0, \\ \phi(z) &= z^{\nu/2}, \qquad \text{for} \ \ z > 1. \end{aligned}$$

We consider nonnegative initial values $a_i^0 \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $1 \le i \le P$, verifying

$$\sup_{1\le i\le P} \left\{ \int_{\mathbb{R}^N} a_i^0(1 + |x| + |\ln(a_i^0)|)\, dx \right\} < \infty, \tag{6}$$

but without smallness condition. We note that these additional constraints correspond to the finite initial mass and total entropy, together with a condition of confinement of the mass near the origin.

The main result of this paper is the following theorem.

Theorem 1 *Let $\nu < 2$. Consider a system (1) verifying hypothesis (2), (3), (4), and (5). Then, for any $a^0 = (a_1^0, \cdots, a_P^0)$, with $a_i^0 \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $a_i^0 \ge 0$ for $1 \le i \le P$, verifying the condition (6), there exists a unique smooth global solution $a = (a_1, \cdots, a_P)$ defined on $[0, \infty) \times \mathbb{R}^N$. Moreover, $a_i \ge 0$ for all $1 \le i \le P$.*

A system of particular interest, which is a limit case (not included) of this study, is the quadratic system of reaction with four species corresponding to the chemical reaction

$$A_1 + A_3 \rightleftharpoons A_2 + A_4.$$

In this case the system (1) has $P = 4$, and

$$Q_i(a) = (-1)^i(a_1a_3 - a_2a_4), \qquad a = (a_1, a_2, a_3, a_4) \in \mathbb{R}_+^4, \ \ 1 \le i \le 4,$$

where a_i is the concentration of the species A_i. Let us denote

$$U_i = \sqrt{a_i}, \qquad 1 \le i \le 4.$$

The mass conservation and entropy dissipation give (see section 2) the following bounds for any $T > 0$:

$$U \in L^\infty(0, T; L^2(\mathbb{R}^N)), \qquad \nabla_x U \in L^2((0, T) \times \mathbb{R}^N).$$

Also, the universal scaling is given by

$$U_\varepsilon(t,x) = \varepsilon U(\varepsilon^2 t, \varepsilon x), \qquad \varepsilon > 0.$$

This means that U is a solution to the system (1) if and only if U_ε is a solution to the same problem. Note that U has exactly the same conserved quantities and the exact same universal scaling as the Navier-Stokes equation, which is supercritical for $N > 2$. We interpret this by saying that our result gives the full regularity for a family of parabolic systems, in any dimension, which has almost the same supercriticality as the Navier-Stokes equation.

However, we observe that these systems of reaction-diffusion have some features quite different from fluid mechanics. In the inviscid case where $D_i = 0$ for all $i,\ 1 \le i \le P$, a maximum principle ensures that the solutions remain globally bounded. But this structure seems to break down in the diffusive case provided that the diffusion coefficients D_i's are not all the same. Surprisingly, such systems could be destabilized by the diffusion.

One of the key arguments in our proof is the use of a local parabolic regularization effect. This argument basically claims that if the nonlinear terms are somehow controlled and small in a cylinder $(t_0-1, t_0) \times B(x_0, 1)$, then the value of the solution at (t_0, x_0) is bounded by 1. This argument is at the heart of the partial regularity results for the Navier-Stokes equations [4, 17, 28, 29]. In [31], a new proof is proposed based on the De Giorgi's parabolic regularity method [7]. This technique is particularly powerful and gives important results in different physical areas, such as the quasigeostrophic equation [5], and the compressible Fourier-Navier-Stokes system [18, 19]. This method has been used for the first time in the context of reaction-diffusion equation in [16]. This paper contains the cases $N \le 2$, even with quadratic growth. It corresponds to the entropy critical case.

The local parabolic regularization results are based on recursive controls, along a family of shrinking cylinders, of the nonlinear terms in a cylinder by the entropy (or energy) on a bigger one. We note that, in the context of reaction-diffusion systems, the classical methods based on the Green function of the heat equation do not work in the supercritical cases. The main problem is that we have different diffusion coefficients. In this context, the De Giorgi method proves to be particularly powerful as it exploits the physical entropy quantity (5). Using this property, this method depletes the exponent of the nonlinearity by 1. This is one of the key facts which allows us to work with subquadratic nonlinearity for any dimension N. It follows that, locally, it is enough to control recursively slightly bigger (to the log) norms that the L^1 norm to get the regularization since such norms are controlled by the entropy.

To get the global regularity, we shrink these local norms, about any point (t_0, x_0) through the universal scaling

$$a^\varepsilon(s,y) = \varepsilon^{\frac{2}{\nu-1}} a(\varepsilon^2 s + t_0, \varepsilon y + x_0), \quad \text{for } \varepsilon > 0.$$

If $a = (a_1, \cdots, a_P)$ is a solution to the system, then $a^\varepsilon = (a_1^\varepsilon, \cdots, a_P^\varepsilon)$ is a solution to a system with a reaction term Q^ε which shares the same properties as the reaction term Q [Hypothesis (2), (3), (4), and (5)] with the same ν and the same constant Λ. If there is an $\varepsilon > 0$, small enough, for which the norm is very small, then we can use the local parabolic regularization result on $a^\varepsilon = (a_1^\varepsilon, \cdots, a_P^\varepsilon)$ to ensure the regularity of $a = (a_1, \cdots, a_P)$ at the point (t_0, x_0). In the entropy supercritical cases, all the quantities obtained through the entropy blow up along the rescaling. But, on the other hand, we can show that the solution is globally bounded in a very weak space which has the same homogeneity as $L^\infty(W^{-2,\infty})$. The norm in this weak space shrinks in the subquadratic growth $\nu < 2$ as

$$\|a^\varepsilon\|_{L^\infty(W^{-2,\infty})} = \varepsilon^{\frac{4-2\nu}{\nu-1}} \|a\|_{L^\infty(W^{-2,\infty})}.$$

It is the nonnegativity of the solutions that allows us to control the depleted nonlinear term by this weak norm. We can use, then, the local parabolic regularization property.

The control of the solution in this weak space is based on a nice duality argument introduced by Pierre and Schmitt [26]. In [26], they obtain a global bound in $L^2(L^2)$ of the solution. A similar argument has been used in [10] to get global weak solutions. The $L^2(L^2)$ norm does not shrink through the universal scaling for the entropy supercritical cases. However, we will show that the same method can be applied for the weak norm that we present.

The study of this weak norm was suggested by the paper [32], where the $L^\infty(W^{-1,\infty})$ norm is shown to play an important role in the Navier-Stokes equation. We observe that the $L^\infty(W^{-1,\infty})$ norm has the same scaling as the $L^\infty(BMO^{-1})$ norm studied by Koch and Tataru for the Navier-Stokes equation. The weak norm that we introduce corresponds, formally, for the quadratic case, to the $L^\infty(W^{-2,\infty})$ norm.

The additional control of the weak norm comes from a linear equation verified by the total mass $\rho = \sum_i^P a_i$. Pierre and Schmitt [26] use this extra equation to get the $L^2(L^2)$ estimates. They also show that the full regularity cannot come solely from this argument as they provide some explicit examples of solutions to this type of linear equation which blow up in finite time.

We note that, in the context of the Navier-Stokes equation, an equation on the vorticity is also provided as an extra. For the Euler equation (the inviscid case), the vorticity ω is a solution to the equation:

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0.$$

If we drop the dependence of vorticity on the velocity u, the above equation can be seen as a linear equation on ω, which can be solved in the Lagrangian coordinates as

$$\omega(t, a) = (\omega(0, a) \cdot \nabla_a) X(t, a),$$

where X is the flow given by

$$\partial_t X(t, a) = u(t, X(t, a)).$$

Note that, for any smooth initial values, this gives a uniform control of the vorticity in $L^\infty(W^{-1,\infty}_{\text{loc}})$ in the Lagrangian coordinates. This norm is, actually, shrinking through the universal scaling of the Navier-Stokes equation. Unfortunately, this structure seems to be destroyed by the viscosity term.

Such property provides another analogy with the reaction-diffusion systems: as for the maximum principle for the reaction-diffusion systems, a crucial supercritical structure known on the Euler system could be destabilized by the diffusion term.

Global existence of weak solutions of (1), was established in [10]. The dissipation property (5) is also the basic tool for studying the asymptotic trend to equilibrium [8, 9] in the spirit of the entropy/entropy dissipation techniques which are presented e.g. in [33] (we refer also to [2] for further investigation of the large time behavior of nonlinear evolution systems using the entropy dissipation). Let us also mention that (1) can be derived through hydrodynamic scaling from kinetic models, see [3].

As remarked above, the results of Theorem 1 are trivial if the diffusion coefficients D_i's are all equal. Another trivial case corresponds to $P = 2$ (two species) where a maximum principle holds (even without the subquadratic property (3)). For the sake of completeness we will give a proof of this result in the appendix as we did not find this case in the literature.

Theorem 2 *Consider a system (1), with $P = 2$, and verifying hypothesis (2) (4), and (5). Then, for any $a^0 = (a_1^0, a_2^0)$, with $a_1^0, a_2^0 \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $a_1^0, a_2^0 \geq 0$, verifying the condition (6), there exists a unique smooth global solution $a = (a_1, a_2)$ defined on $[0, \infty) \times \mathbb{R}^N$. Moreover, $a_1, a_2 \geq 0$, and*

$$\sup_{(t,x)\in(0,\infty)\times\mathbb{R}^N} \{a_1(t, x), a_2(t, x)\} \leq \sup_{x\in\mathbb{R}^N} \{a_1^0(x), a_2^0(x)\}.$$

The proof of the maximum principle collapses for $P \geq 3$.

Some of the estimates in this paper will be based on results obtained in [16], but, for the sake of completeness this paper is self contained.

For any smooth initial data, standard theory ensures the existence of a smooth solution on (at least) a short time $[0, T)$, $T > 0$. We set T_0 to be the biggest of such lapse of time. Our aim is to show that $T_0 = \infty$. Standard bootstrapping arguments give that if $T_0 < \infty$ then

$$\lim_{t\to T_0} \sup_{1\leq i\leq P} \|a_i(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

We will obtain a uniform bound on $[0, T_0)$ contradicting the blow-up of the solution in finite time. Indeed, we will show that this bound depends only on T_0 and the quantity M_0 defined by

$$M_0 = \sup_{1\leq i\leq P} \left\{ \int_{\mathbb{R}^N} a_i^0(1 + |x| + |\ln(a_i^0)|)\, dx + \|a_i^0\|_{L^\infty(\mathbb{R}^N)} \right\}. \tag{7}$$

In the second section, we provide standard a priori estimates based on the mass conservation and the entropy dissipation. The third section is dedicated to the local parabolic regularization principle. The duality arguments are given in the fourth section. We introduce and then apply the rescaling arguments in the last section.

2 Entropy Dissipation

In this section, we derive a priori estimates on $[0, T_0)$ where the solution is smooth. The dimensional cases of $N \leq 2$ have been already studied in [16], but they can also be deduced from the techniques that we present in this paper. (The reader can easily replace the classical inequalities in the sequel of the proofs for dimensions $N \leq 2$). Henceforth, we will analyze here only the dimensions $N \geq 3$.

We discuss the a priori estimates that can be naturally deduced from (4) and (5).

Proposition 3 *There exist two constants C_0 and C_1, such that the following is true. Let (1) be any system verifying (2), (4), and (5), and any initial values $a_i^0 \geq 0$, smooth, verifying*

$$M_0 = \sup_{1 \leq i \leq P} \left(\|a_i^0\|_{L^\infty(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_i^0 \big(1 + |x| + |\ln(a_i^0)|\big)\, dx \right) < \infty. \tag{8}$$

Let $a = (a_1, \cdots, a_P)$ be the associated solution. Assume that a is regular on its maximal lapse of time $[0, T_0)$. Let

$$\mathfrak{D}(a) = -\sum_{i=1}^P Q_i(a) \ln(a_i) \geq 0.$$

Then, for any $0 < T < T_0$,

$$\sup_{0 \leq t \leq T} \left\{ \sum_{i=1}^P \int_{\mathbb{R}^N} a_i \big(1 + |x| + |\ln(a_i)|\big)(t, x)\, dx \right\} + 2\underline{d} \sum_{i=1}^P \int_0^T \int_{\mathbb{R}^N} \left|\nabla_x \sqrt{a_i}\right|^2 dx\, ds$$
$$+ \sum_{i=1}^P \int_0^T \int_{\mathbb{R}^N} \mathfrak{D}(a) dx\, ds \leq (C_0 + C_1 T)(M_0 + 1).$$

Proof. First, thanks to (2), we have $a_i(t, x) \geq 0$ for any $t \in [0, T_0)$, $x \in \mathbb{R}^N$, and $1 \leq i \leq P$. As a consequence of (4) and (5), we get

$$\frac{d}{dt} \sum_{i=1}^P \int_{\mathbb{R}^N} a_i\, dx = 0$$

and

$$\frac{d}{dt}\left(\sum_{i=1}^{P}\int_{\mathbb{R}^N} a_i\big(1+\ln(a_i)\big)\,dx\right)+\sum_{i=1}^{P}\int_{\mathbb{R}^N} D_i\nabla_x a_i\cdot\frac{\nabla_x a_i}{a_i}\,dx+\int_{\mathbb{R}^N}\mathfrak{D}\,dx=0.$$

Also,

$$\sum_{i=1}^{P}\int_{\mathbb{R}^N} D_i\nabla_x a_i\cdot\frac{\nabla_x a_i}{a_i}\,dx\geq \underline{d}\sum_{i=1}^{P}\int_{\mathbb{R}^N}\frac{|\nabla_x a_i|^2}{a_i}\,dx=4\underline{d}\sum_{i=1}^{P}\int_{\mathbb{R}^N}\left|\nabla_x\sqrt{a_i}\right|^2dx.$$

Moreover

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{P}\int_{\mathbb{R}^N} a_i|x|\,dx &= -\sum_{i=1}^{P}\int_{\mathbb{R}^N} D_i\nabla_x a_i\cdot\frac{x}{|x|}\,dx\\ &\leq \bar{d}\sum_{i=1}^{P}\int_{\mathbb{R}^N}|\nabla_x a_i|\,dx=\bar{d}\sum_{i=1}^{P}\int_{\mathbb{R}^N}\frac{|\nabla_x a_i|}{\sqrt{a_i}}\sqrt{a_i}\,dx\\ &\leq \frac{\underline{d}}{2}\sum_{i=1}^{P}\int_{\mathbb{R}^N}\frac{|\nabla_x a_i|^2}{a_i}\,dx+\frac{\bar{d}^2}{2\underline{d}}\sum_{i=1}^{P}\int_{\mathbb{R}^N} a_i\,dx.\end{aligned}$$

Let $0\leq t\leq T$. We integrate on the interval $[0,t]$,

$$\begin{aligned}&\sum_{i=1}^{P}\int_{\mathbb{R}^N} a_i\big(1+|x|+\ln(a_i)\big)\,dx+\frac{\underline{d}}{2}\sum_{i=1}^{P}\int_0^t\int_{\mathbb{R}^N}\frac{|\nabla_x a_i|^2}{a_i}\,dx\,ds+\int_0^t\int_{\mathbb{R}^N}\mathfrak{D}\,dx\,ds\\ &\qquad\leq M_0+\frac{\bar{d}^2}{2\underline{d}}\sum_{i=1}^{P}\int_0^t\int_{\mathbb{R}^N} a_i\,dx\,ds\\ &\qquad\leq \big(1+t\bar{d}^2/(2\underline{d})\big)M_0.\end{aligned}$$

Finally, we estimate the negative part of the $a_i\ln(a_i)$:

$$\begin{aligned}\int_{\mathbb{R}^N} a_i|\ln(a_i)|\,dx &= \int_{\mathbb{R}^N} a_i\ln(a_i)\,dx-2\int_{\mathbb{R}^N} a_i\ln(a_i)\big(\mathbb{1}_{0\leq a_i\leq e^{-|x|/2}}+\mathbb{1}_{e^{-|x|/2}\leq a_i\leq 1}\big)\,dx\\ &\leq \int_{\mathbb{R}^N} a_i\ln(a_i)\,dx+\frac{4}{e}\int_{\mathbb{R}^N} e^{-|x|/4}\,dx+\int_{\mathbb{R}^N}|x|a_i\,dx\end{aligned}$$

since $-s\ln(s)\leq\frac{2}{e}\sqrt{s}$ for any $0\leq s\leq 1$. Combining together all the pieces we conclude the statement of the proposition.

∎

3 Local parabolic regularization principle

We consider solutions of (1) that are defined for negative times. The following proposition is the main result of this section.

Proposition 4 *Let (1) be a system verifying (2), (3), (4), and (5). Let* $a = (a_1, \cdots, a_P)$ *be a solution to (1) on* $(-3, 0) \times B(0, 3)$*, such that* a_i *are nonnegative functions for* $1 \leq i \leq P$*. Then, for any* $p > 1$*, there exists a universal constant* $\delta^\star > 0$ *(depending on the constant* Λ *as in (3),* $\bar{d}$*,* $\underline{d}$*,* P*,* N *and* p*), such that, if* $a = (a_1, \cdots, a_P)$ *verifies*

$$\sum_{i=1}^{P} \|a_i\|_{L^p((-3,0)\times B(0,3))} \leq \delta^\star,$$

then, $0 \leq a_i(0, 0) \leq 1$ *for* $1 \leq i \leq P$*.*

The value at $(0, 0)$ can be hence controlled by any L^p norm, $p > 1$, on a surrounding cylinder. Such results are quite surprising, since we work with subquadratic reaction terms.

In the spirit of the Stampacchia cut-off method, L^∞ bounds of solutions of certain PDEs can be deduced from the behavior of suitable non linear functionals. Here, such functionals are constructed in a way that they use the dissipation property (5). This is based on the De Giorgi techniques and it is reminiscent of the method introduced by Alikakos [1].

Let us consider the non negative, C^1, and convex function

$$\Phi(z) = \begin{cases} (1+z)\ln(1+z) - z & \text{if } z \geq 0, \\ 0 & \text{if } z \leq 0. \end{cases}$$

We study the evolution of the "entropy at level R"

$$\sum_{i=1}^{P} \int_{\mathcal{B}_n} \Phi(a_i - R)\, dx,$$

for $R \geq 0$, where the set $\mathcal{B}_n$ will be specified later.

Lemma 5 *Let* $1 < \nu < 2$*. For any* $a \in \mathbb{R}^P$*, and for any* $0 \leq R \leq 1$*, we have*

$$\sum_{i=1}^{P} |Q_i(a) - Q_i(1 + [a - R]_+)| \leq 2P\Lambda|1 + [a - R]_+|^{\nu-1}$$

where Λ *is given by (3) and* $1 + [a - R]_+ := (1 + [a_1 - R]_+, \cdots, 1 + [a_P - R]_+)$*.*

Proof. For $|u| \le |v|$, by the sub-quadratic growth for the reaction term Q, hypothesis (3), we have

$$\sum_{i=1}^{P} |Q_i(u) - Q_i(v)| \le \Lambda P |v-u||v|^{\nu-1}.$$

The inequality follows by substituting (u,v) by $(a, 1+[a-R]_+)$. ∎

Let us set $k_n = 1 - 1/2^n$, $t_n = 1 + 1/2^n$, $\mathcal{B}_n = B(0,t_n)$, $\mathcal{Q}_n = (-t_n, 0) \times \mathcal{B}_n$. Note that $\mathcal{B}_n \subset \mathcal{B}_{n-1}$ and $\mathcal{Q}_n \subset \mathcal{Q}_{n-1}$.

We introduce the cut-off functions

$$\begin{cases} \zeta_n : \mathbb{R}^N \to \mathbb{R}, & 0 \le \zeta_n(x) \le 1, \\ \zeta_n(x) = 1 \text{ for } x \in \mathcal{B}_n, & \zeta_n(x) = 0 \text{ for } x \in \complement \mathcal{B}_{n-1}, \\ \sup\limits_{i,j \in \{1,\dots,N\},\ x \in \mathbb{R}^N} |\partial^2_{ij} \zeta_n(x)| \le C\ 2^{2n} \text{ with } C \text{ universal constant.} \end{cases}$$

Next, we give an estimate on the local dissipation of entropy at the level R.

Proposition 6 *There exists a universal constant $\hat{C}$ (depending on $\bar{d}$,$\underline{d}$, Λ and P), such that for every $a = (a_1, \cdots, a_P)$ solution of (1), for any $0 \le R \le 1$, we have*

$$\sup_{-t_n \le t \le 0} \Big\{ \sum_{i=1}^{P} \int_{\mathcal{B}_n} \Phi(a_i - R)(t,x)\, dx \Big\} + \underline{d} \sum_{i=1}^{P} \int\int_{\mathcal{Q}_n} |\nabla_x \sqrt{1 + [a_i - R]_+}|^2 \, dx\, d\tau$$

$$\le 2^{2n} \hat{C} \sum_{i=1}^{P} \int\int_{\mathcal{Q}_{n-1}} (1 + [a_i - R]_+) \ln(1 + [a_i - R]_+)\, dx\, d\tau.$$

Proof. We multiply (1) by $\zeta_n \Phi'(a_i - R)$, and we sum

$$\frac{d}{dt} \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} \zeta_n \Phi(a_i - R)\, dx = A + B \tag{9}$$

where

$$A := \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} D_i \Delta a_i\ \Phi'(a_i - R) \zeta_n \, dx$$

and

$$B := \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} Q_i(a) \Phi'(a_i - R) \zeta_n \, dx.$$

We rewrite A as

$$A = -E + F$$

where

$$E := \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} D_i |\nabla a_i|^2 \ \Phi''(a_i - R)\zeta_n \, dx$$

and

$$F := \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} D_i \Phi(a_i - R)\Delta\zeta_n \, dx.$$

Moreover,

$$\begin{aligned}
E &= \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} D_i |\nabla_x a_i|^2 \ \Phi''(a_i - R)\zeta_n \, dx = \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} D_i \nabla_x a_i \cdot \nabla_x a_i \ \frac{\mathbb{1}_{a_i \geq R}}{1 + [a_i - R]_+} \, dx \\
&= \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} D_i \nabla_x (1 + [a_i - R]_+) \cdot \nabla_x (1 + [a_i - R]_+) \ \frac{dx}{1 + [a_i - R]_+} \\
&\geq \underline{d} \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} \frac{\left|\nabla_x (1 + [a_i - R]_+)\right|^2}{1 + [a_i - R]_+} \, dx \\
&\geq 4\underline{d} \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} \left|\nabla_x \sqrt{1 + [a_i - R]_+}\right|^2 dx.
\end{aligned}$$

Next, we estimate the quantity B from (9). For $0 < \nu \leq 1$,

$$B \leq \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} (1 + [a_i - R]_+) \ln(1 + [a_i - R]_+) \, dx,$$

while, for $1 < \nu < 2$, to get rid of the nonlinearity given by the factor $Q_i(a)$, we rewrite B as

$$\begin{aligned}
&\sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} Q_i(a) \ \ln(1 + [a_i - R]_+) \, dx \\
&= \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} \big(Q_i(a) - Q_i(1 + [a - R]_+)\big) \ \ln(1 + [a_i - R]_+) \, dx \\
&\qquad + \sum_{i=1}^{P} \int_{\mathcal{B}_{n-1}} Q_i(1 + [a - R]_+) \ln(1 + [a_i - R]_+) \, dx.
\end{aligned}$$

By assumption (5) the last term is non positive, while $\big(Q_i(a) - Q_i(1+[a-R]_+)\big)\ \ln(1+[a_i-R]_+)$ can be estimated via Lemma 5. Finally, from (9) and the above, we get

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{P}\int_{\mathcal{B}_{n-1}} \zeta_n \Phi(a_i - R)\,dx + 4\underline{d}\sum_{i=1}^{P}\int_{\mathcal{B}_n} \left|\nabla_x\sqrt{1+[a_i-R]_+}\right|^2 dx \\
&\qquad \le 2^{2n}\hat{C}\left(\sum_{i=1}^{P}\int_{\mathcal{B}_{n-1}} (1+[a_i-R]_+)\ln(1+[a_i-R]_+)\,dx + \sum_{i=1}^{P}\int_{\mathcal{B}_{n-1}} \Phi(a_i-R)\,dx\right).
\end{aligned}$$

To conclude, we integrate on (s,t), for $-t_n < t < 0$, and average on $s \in (-t_{n-1}, -t_n)$. ∎

Let us introduce the following easy lemma.

Lemma 7 *Let*

$$\Psi(z) = (\sqrt{1+z} - 1)\mathbf{1}_{\{z>0\}}, \qquad z \in \mathbb{R}. \tag{10}$$

The function Ψ is nondecreasing, Lipschitz on $\mathbb{R}$, and there exists a constant $C > 0$ such that

$$\Psi(z) \le \tilde{C}\sqrt{\Phi(z)}, \qquad z \in \mathbb{R}, \tag{11}$$

$$\mathbf{1}_{\{\Psi(z-k_n)>0\}} \le \tilde{C}2^n\Psi(z-k_{n-1}), \qquad n \ge 1. \tag{12}$$

Proof. To get (11), we expand Ψ and Φ both at 0 and $+\infty$ to find that there exists $C_1, C_2 > 0$ with

$$\begin{aligned}
\Psi(z)^2 &\le C_1 \inf(z, z^2) \qquad z > 0, \\
\Phi(z) &\ge C_2 \inf(z, z^2) \qquad z > 0.
\end{aligned}$$

To get (12), first consider $z > 2$:
For any $n \ge 1$

$$2^n\Psi(z-k_{n-1}) \ge \Psi(z-k_{n-1}) \ge \Psi(1) \ge (\sqrt{2}-1)\mathbf{1}_{\{\Psi(z-k_n)>0\}}.$$

Next, if $z \le 2$, $\Psi(z-k_n) > 0$ implies that $z > k_n$ and so $z - k_{n-1} \ge 2^{-n}$. Hence, for such z:

$$\begin{aligned}
\Psi(z-k_{n-1}) &\ge [\Psi(z-k_{n-1}) - \Psi(z-k_n)] + \Psi(z-k_n) \\
&\ge \Psi(z-k_{n-1}) - \Psi(z-k_n) \\
&\ge \frac{k_n - k_{n-1}}{\sqrt{1+(z-k_{n-1})_+} + \sqrt{1+(z-k_n)_+}} \ge \frac{2^{-n}}{2\sqrt{3}}.
\end{aligned}$$

∎

We, now, define the sequence $\mathcal{U}_n$ which plays a key role for the proof of the uniform boundedness of the solution.

$$\mathcal{U}_n = \sup_{-t_n \le t \le 0} \sum_{i=1}^{P} \int_{\mathcal{B}_n} \Phi(a_i - k_n)\,dx + \sum_{i=1}^{P} \int\int_{\mathcal{Q}_n} |\nabla_x \Psi(a_i - k_n)|^2\,dx\,ds.$$

Note that, since Ψ is Lipschitz,

$$\nabla_x \Psi(a_i - k_n) = \nabla_x \sqrt{1 + [a_i - k_n]_+}.$$

The following nonlinear estimate is a crucial step for establishing our results.

Lemma 8 *There exists a universal constant C (depending only on Λ, $\bar{d}$, $\underline{d}$, P, and N) such that*

$$\mathcal{U}_n \le C^n \mathcal{U}_{n-1}^{\frac{N+2}{N}}$$

for any $n \ge 1$. Especially, there exists $\delta > 0$ (depending only on Λ, $\bar{d}$, $\underline{d}$, P, and N) such that if $\mathcal{U}_0 \le \delta$ then $\lim_{n\to\infty} \mathcal{U}_n = 0$.

Proof. From Proposition 6 we get the inequality:

$$\begin{aligned}\mathcal{U}_n &\le \hat{C} 2^{2n} \sum_{i=1}^{P} \int\int_{\mathcal{Q}_{n-1}} \ln(1 + [a_i - k_n]_+)(1 + [a_i - k_n]_+)\,dx\,d\tau \\ &\le \hat{C} 2^{2n} \sum_{i=1}^{P} \int\int_{\mathcal{Q}_{n-1}} \mathbf{1}_{\{a_i \ge k_n\}} (1 + [a_i - k_n]_+)^{\frac{N+2}{N}}\,dx\,d\tau.\end{aligned}$$

Hence

$$\mathcal{U}_n \le 2\hat{C} 2^{2n} \left(\sum_{i=1}^{P} \int\int_{\mathcal{Q}_{n-1}} \mathbf{1}_{\{\Psi(a_i - k_n) > 0\}}\,dx\,d\tau + \|\Psi(a_i - k_n)\|^{2\frac{N+2}{N}}_{L^{2\frac{N+2}{N}}(\mathcal{Q}_{n-1})} \right). \tag{13}$$

Next, using (12), we find that

$$\begin{aligned}&\sum_{i=1}^{P} \int\int_{\mathcal{Q}_{n-1}} \mathbf{1}_{\{\Psi(a_i - k_n) > 0\}}\,dx\,d\tau \\ &\le (\tilde{C} 2^n)^{2\frac{N+2}{N}} \sum_{i=1}^{P} \int\int_{\mathcal{Q}_{n-1}} [\Psi(a_i - k_{n-1})]^{2\frac{N+2}{N}}\,dx\,d\tau.\end{aligned}$$

Since $\Psi(a_i - k_n) \le \Psi(a_i - k_{n-1})$, we finally get

$$\mathcal{U}_n \leq 2\hat{C}2^{2n}(\tilde{C}2^n)^{2\frac{N+2}{N}} \sum_{i=1}^{P} \|\Psi(a_i - k_{n-1})\|^{2\frac{N+2}{N}}_{L^{2\frac{N+2}{N}}(\mathcal{Q}_{n-1})}.$$

Using (11), we find that

$$\|\Psi(a_i - k_{n-1})\|^2_{L^\infty(-t_{n-1},0;L^2(\mathcal{B}_{n-1}))} \leq \tilde{C}\|\Phi(a_i - k_{n-1})\|_{L^\infty(-t_{n-1},0;L^1(\mathcal{B}_{n-1}))} \leq \mathcal{U}_{n-1}.$$

By Sobolev imbedding, we find

$$\|\Psi(a_i - k_{n-1})\|^2_{L^2(-t_{n-1},0;L^{\frac{2N}{N-2}}(\mathcal{B}_{n-1}))} \leq c\,\mathcal{U}_{n-1}\ .$$

By interpolation, this gives

$$\|\Psi(a_i - k_{n-1})\|^2_{L^{\frac{2(N+2)}{N}}(\mathcal{Q}_{n-1})} \leq c\mathcal{U}_{n-1}, \text{ with } c \text{ universal constant.}$$

Finally, we get

$$\mathcal{U}_n \leq C^n \mathcal{U}_{n-1}^{\frac{(N+2)}{N}}$$

where C depends only on the constants Λ as in (3), $\bar{d}$, $\underline{d}$, P, and N. ∎

We now discuss the smallness of $\mathcal{U}_0$.

Lemma 9 *For any $p > 1$ there exists a universal constant $C > 0$ (depending only on $\bar{d}$, $\underline{d}$ and p) such that*

$$\mathcal{U}_0 \leq C\big(\sum_{i=1}^{P} \|a_i\|^p_{L^p((-3,0)\times B(0,3))} + \sum_{i=1}^{P} \|a_i\|^{1/2}_{L^p((-3,0)\times B(0,3))}\big).$$

Proof. By definition

$$\mathcal{U}_0 = \sup_{-2\leq t\leq 0} \sum_{i=1}^{P} \int_{B(0,2)} \Phi(a_i)\,dx + \int_{-2}^{0}\int_{B(0,2)} \left|\nabla_x\sqrt{a_i+1}\right|^2 dx\,d\tau.$$

We will use the following facts:

$$\Phi(a_i) \leq C(a_i(1 + |\ln(a_i)|); \tag{14}$$

$$\int_{-2}^{0}\int_{B(0,2)} \left|\nabla_x\sqrt{a_i+1}\right|^2 dx\,d\tau \leq \int_{-2}^{0}\int_{B(0,2)} \left|\nabla_x\sqrt{a_i}\right|^2 dx\,d\tau. \tag{15}$$

The mass conservation yields

$$\frac{d}{dt}\sum_{i=1}^{P}\int_{B(0,3)} \zeta_0(x) a_i(t,x)\,dx = \sum_{i=1}^{P}\int_{B(0,3)} D_i\Delta\zeta_0(x)\ a_i(t,x)\,dx \le C\sum_{i=1}^{P}\int_{B(0,3)} a_i(t,x)\,dx,$$

where $\zeta_0 \ge 0$, $\zeta_0 \equiv 1$ on $B(0,2)$, has bounded second order derivatives and it is supported on $B(0,3)$. Let $t \in (-2,0)$. Let $\tau \in (-3,t)$. We integrate over the time interval (τ,t), and then we average over $\tau \in (-3,-2)$. Hence, we get

$$\sup_{-2\le t\le 0}\sum_{i=1}^{P}\int_{B(0,2)} a_i(t,x)\,dx \quad\le\quad C\sum_{i=1}^{P}\int_{-3}^{0}\int_{B(0,3)} a_i(\tau,x)\,dx\,d\tau.$$

Similarly, the entropy dissipation yields

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{P}\int_{B(0,3)} &\zeta_0(x)\ a_i\ln(a_i)\,dx + \int_{B(0,3)}\zeta_0(x)\frac{D_i|\nabla_x a_i|^2}{a_i}\,dx\\ &\le \sum_{i=1}^{P}\int_{B(0,3)} D_i\Delta\zeta_0(x)\ a_i\ln(a_i)\,dx\\ &\le C\sum_{i=1}^{P}\int_{B(0,3)} a_i|\ln(a_i)|\,dx.\end{aligned}$$

Again we integrate with respect to the time variable. We shall also use the trick

$$u|\ln(u)| = u\ln(u) - 2u\ln(u)\mathbb{1}_{0\le u\le 1} \le u\ln(u) + C\sqrt{u}.$$

It follows that

$$\begin{aligned}\sup_{-2\le t\le 0}\sum_{i=1}^{P}\int_{B(0,2)} a_i|\ln(a_i)|\,dx &+ \underline{d}\sum_{i=1}^{P}\int_{-2}^{0}\int_{B(0,2)}\left|\nabla_x\sqrt{a_i}\right|^2 dx\,d\tau\\ &\le C\sum_{i=1}^{P}\left(\int_{-3}^{0}\int_{B(0,3)}\left(a_i|\ln(a_i)| + \sqrt{a_i}\right)\,dx\,d\tau\right).\end{aligned}$$

Combining (15), (14), and the definition of $\mathcal{U}_0$ yields to the desired inequality. ∎

The proof of Proposition 4 follows from Lemma 8 and Lemma 9.

4 Duality arguments

In this section we derive a uniform bound on the solutions $a = (a_1, \cdots, a_P)$ in a weak norm. We will show that this weak norm shrinks through the universal scaling. The proof relies on a nice duality argument first used in [26].

Let $a = (a_1, \cdots, a_P)$ be a solution to the system (1) with initial data $a^0 = (a_1^0, \cdots, a_P^0)$. Then, the function $\rho = \sum_{i=1}^P a_i$ is a solution to the Cauchy problem

$$\begin{cases} \partial_t \rho - \Delta(d\rho) = 0, \qquad t > 0,\ x \in \mathbb{R}^N, \\ \rho^0(x) = \rho(0, x) = \displaystyle\sum_{i=1}^P a_i^0, \qquad x \in \mathbb{R}^N, \end{cases} \tag{16}$$

with

$$d(t, x) = \frac{\sum_{i=1}^P D_i a_i}{\sum_{i=1}^P a_i}.$$

Note that the diffusion coefficient $d(t, x)$ is elliptic. Indeed, it is bounded from above and below:

$$0 < \underline{d} \leq d(t, x) \leq \bar{d}. \tag{17}$$

The equation in (16) seems so to be a nice parabolic equation, except that it is not in the standard divergence or non divergence form. In the non divergence form, the equation would provide the maximum principle. In the classical divergence form, De Giorgi showed in [7] that such solutions are bounded locally in C^α. Surprisingly, the behavior of solutions of parabolic equations written as (16) is very different. Note that we do not have a priori bounds based on the regularity of d. In [26], Pierre and Schmitt show that any solution of the parabolic equation (16), with (17) (but no assumption on the regularity of d), and with regular enough initial values, lies in $L^2((0, T_0) \times \mathbb{R}^N)$ for $T_0 > 0$. However, they also give explicit examples of solutions which blow-up in $L^p((0, T_0) \times \mathbb{R}^N)$ for some $p > 2$. It follows that global regularity of solutions to (1) cannot rely only on the equation solved by ρ.

As usual we denote $\mathcal{D}(\mathbb{R}^N) := \mathcal{C}_0^\infty(\mathbb{R}^N)$. Let us first introduce our weak space.

Definition 10 *We define the space $\mathcal{L}_w(\mathbb{R}^N)$ as the dual space of $\{\rho \in \mathcal{D}(\mathbb{R}^N) : \Delta\rho \in L^1\}$:*

$$\mathcal{L}_w(\mathbb{R}^N) := \{f \in \mathcal{D}'(\mathbb{R}^N) : ||f||_{\mathcal{L}_w(\mathbb{R}^N)} < \infty\}$$

where

$$||f||_{\mathcal{L}_w(\mathbb{R}^N)} := \sup_{\psi \in \mathcal{D}(\mathbb{R}^N), ||\Delta\psi||_{L^1} \leq 1} |< f, \psi >| \,.$$

The following proposition is the main result of this section.

Proposition 11 *Let $a = (a_1, \cdots, a_P)$ be a smooth and bounded solution of (1) on the time interval $[0, T]$ for any $T < T_0$ (with possible blow-up at $T = T_0$). Then $\rho = \sum_{i=1}^{P} a_i$ verifies*

$$\rho \in L^\infty(0, T_0; \mathcal{L}_w(\mathbb{R}^N)),$$

with,

$$\|\rho(T)\|_{\mathcal{L}_w(\mathbb{R}^N)} \leq \|\rho^0\|_{\mathcal{L}_w(\mathbb{R}^N)}, \quad \textit{for all } T < T_0. \tag{18}$$

Proof. Let us fix $T < T_0$. The proof consists of two steps:
Step 1: Since $a = (a_1, \cdots, a_P)$ is smooth, the function d is smooth also, except possibly at the points (t, x) where $\rho = 0$. For small $\mu > 0$, let d_μ denote a smooth approximation of d verifying

$$\|d_\mu\|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq \overline{d}, \qquad d_\mu(t, x) = d(t, x) \;\text{ if }\; \rho(t, x) \geq \mu. \tag{19}$$

Let ρ_μ be a solution to the problem:

$$\begin{array}{l} \partial_t \rho_\mu - \Delta(d_\mu \rho_\mu) = 0, \;\; x \in \mathbb{R}^N, \;\; 0 < t < T, \\ \rho(0, \cdot) = \rho_\mu(0) \in \mathcal{D}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N). \end{array} \tag{20}$$

We claim that for such a solution, we have for every $0 \leq t \leq T$:

$$\|\rho_\mu(t)\|_{L^1(\mathbb{R}^N)} \leq \|\rho_\mu(0)\|_{L^1(\mathbb{R}^N)}. \tag{21}$$

To show this claim, consider the dual problem, for any $0 < \overline{T} < T$:

$$\begin{array}{l} \partial_t \phi + d_\mu \Delta \phi = 0, \qquad x \in \mathbb{R}^N, 0 \leq t \leq \overline{T}, \\ \phi(\overline{T}) = \phi_{\overline{T}} \in \mathcal{D}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{array} \tag{22}$$

The maximum principle, applied to the problem (22), gives a uniform bound for the L^∞ norm of ϕ:

$$\|\phi(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\phi_{\overline{T}}\|_{L^\infty(\mathbb{R}^N)},$$

for any $0 < t < \overline{T}$. But, for every $0 < t < \overline{T}$,

$$\frac{d}{dt} \int_{\mathbb{R}^N} \rho_\mu(t, x) \phi(t, x)\, dx = 0.$$

Then, for any $\phi_{\overline{T}} \in \mathcal{D}(\mathbb{R}^N)$ with $\|\phi_{\overline{T}}\|_{L^\infty(\mathbb{R}^N)} \leq 1$, we have $\|\phi(0)\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \rho_\mu(\overline{T}, x) \phi_{\overline{T}}(x)\, dx \right| &= \left| \int_{\mathbb{R}^N} \rho_\mu(0, x) \phi(0, x)\, dx \right| \\ &\leq \|\rho_\mu(0)\|_{L^1(\mathbb{R}^N)} \|\phi(0)\|_{L^\infty(\mathbb{R}^N)} \\ &\leq \|\rho_\mu(0)\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

This proves claim (21).

Step 2: We consider, now, solutions to the following dual problem

$$\begin{aligned}&\partial_t \phi + d_\mu \Delta\phi = 0, \qquad x \in \mathbb{R}^N,\ 0 \le t \le T,\\ &\phi(T) = \phi_T \in \mathcal{D}(\mathbb{R}^N),\end{aligned} \tag{23}$$

for any ϕ_T such that

$$\|\Delta\phi_T\|_{L^1(\mathbb{R}^N)} \le 1.$$

The function

$$\rho_\phi(t,x) = \Delta\phi(T-t,x), \qquad 0 \le t \le T, x \in \mathbb{R}^N,$$

verifies (20) with initial value $\Delta\phi_T$. Hence, (21) ensures that

$$\|\Delta\phi(t)\|_{L^1(\mathbb{R}^N)} \le \|\Delta\phi_T\|_{L^1(\mathbb{R}^N)} \le 1 \qquad 0 \le t \le T. \tag{24}$$

But, for any solution $a = (a_1, \cdots, a_P)$ of (1), $\rho = \sum_{i=1}^P a_i$ verifies (16) and so we have:

$$\frac{d}{dt}\int_{\mathbb{R}^N} \rho(t,x)\phi\, dx = \int_{\mathbb{R}^N} \rho(t,x)(d-d_\mu)\Delta\phi\, dx.$$

Integrating in time on $[0,T]$, we find

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \phi_T(x)\rho(T,x)\, dx\right| &\le \left|\int_{\mathbb{R}^N} \phi(0,x)\rho^0(x)\, dx\right| + \left|\int\int_{\mathbb{R}^N} \rho(d-d_\mu)\Delta\phi\, dx\, dt\right|\\ &\le \|\rho^0\|_{\mathcal{L}_w(\mathbb{R}^N)} + 2T\mu\bar{d}\|\Delta\phi\|_{L^\infty(0,T;L^1(\mathbb{R}^N))}\\ &\le \|\rho^0\|_{\mathcal{L}_w(\mathbb{R}^N)} + 2T\mu\bar{d}.\end{aligned}$$

In the second line we have used the definition of the weak norm together with (24), and (19). In the last line we have used again (24).

Passing to the limit, as $\mu \to 0$, for any $\phi_T \in \mathcal{D}(\mathbb{R}^N)$ such that $\|\Delta\phi_T\|_{L^1(\mathbb{R}^N)} \le 1$, we have

$$\left|\int_{\mathbb{R}^N} \phi_T(x)\rho(T,x)\, dx\right| \le \|\rho^0\|_{\mathcal{L}_w(\mathbb{R}^N)}.$$

This implies, thanks to the definition of the weak norm that:

$$\|\rho(T)\|_{\mathcal{L}_w(\mathbb{R}^N)} \le \|\rho^0\|_{\mathcal{L}_w(\mathbb{R}^N)}.$$

The result follows, since this holds for any $T < T_0$.

∎

As a consequence of this proposition we have the following uniform estimate on the $\mathcal{L}_w(\mathbb{R}^N)$ norm of the total mass.

Corollary 12 *There exists a universal constant C, such that, for any $a = (a_1, \cdots, a_P)$ regular and bounded solution to (1) on $[0,T] \times \mathbb{R}^N$ for any $T < T_0$ (with possible blow-up at T_0), we have $\rho \in L^\infty(0,T_0; \mathcal{L}_w(\mathbb{R}^N))$, and for any $t < T_0$*

$$\|\rho(t)\|_{\mathcal{L}_w(\mathbb{R}^N)} \le C \left(\|\rho^0\|_{L^\infty(\mathbb{R}^N)} + \|\rho^0\|_{L^1(\mathbb{R}^N)} \right).$$

Proof. In the view of Proposition 11, we only need to show that the weak norm can be controlled by stronger norms. We set

$$\phi = \Gamma * \rho^0$$

where $\Gamma(x) = \frac{1}{N(N-2)\omega_N} \frac{1}{|x|^{N-2}}$ (ω_N is the volume of the unit ball in $\mathbb{R}^N$); then, the function ϕ solves

$$-\Delta \phi = \rho^0.$$

Since $\Gamma \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, there exists a constant C such that

$$\|\phi\|_{L^\infty(\mathbb{R}^N)} \le C(\|\rho^0|\|_{L^1(\mathbb{R}^N)} + \|\rho^0\|_{L^\infty(\mathbb{R}^N)}).$$

Therefore, for any $\psi \in \mathcal{D}(\mathbb{R}^N)$, with $\|\Delta\psi\|_{L^1(\mathbb{R}^N)} \le 1$, we have

$$|\int_{\mathbb{R}^N} \rho^0 \psi| = |\int_{\mathbb{R}^N} \Delta\phi\psi| = |\int_{\mathbb{R}^N} \phi\Delta\psi| \le \|\phi\|_{L^\infty(\mathbb{R}^N)} \|\Delta\psi\|_{L^1(\mathbb{R}^N)} \le C(\|\rho^0\|_{L^1(\mathbb{R}^N)} + \|\rho^0\|_{L^\infty(\mathbb{R}^N)}).$$

∎

We observe that the nonnegativity of the mass ρ allows us to control stronger norms from this weak norm.

Lemma 13 *Let f be a nonnegative function on $\mathbb{R}^N$ such that $f \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, then for any compact subset K of $\mathbb{R}^N$, $K \subset\subset \mathbb{R}^N$, there is a positive constant $C(K)$ such that*

$$\|f\|_{L^1(K)} \le C(K) \|f\|_{\mathcal{L}_w(\mathbb{R}^N)}. \tag{25}$$

Proof. Let K be a compact set in $\mathbb{R}^N$ such that $K \subset \mathcal{B}(0,r)$, where $\mathcal{B}(x_0,r)$ is a ball centered at some $x_0 \in K$ of radius $r > 0$. Let $\phi \in C^\infty(\mathbb{R}^N)$ be a smooth nonnegative function, such that $\phi = 1$ on K, and $\phi = 0$ on $\complement(\mathcal{B}(x_0, 2r))$. Set

$$C(K) = \|\Delta\phi\|_{L^1(\mathbb{R}^N)}.$$

Then, the L^1 norm of the Laplacian of $\dfrac{\phi}{C(K)}$ is equal to 1, which implies that

$$\|f\|_{L^1(K)} \le \int_{\mathbb{R}^N} \phi(x) f(x)\, dx \le C(K) \int_{\mathbb{R}^N} \frac{\phi(x)}{C(K)} f(x)\, dx \le C(K) \|f\|_{\mathcal{L}_w(\mathbb{R}^N)}.$$

∎

As a consequence of this lemma we have the following corollary.

Corollary 14 *For any integer $P > 0$, and any $p \in (1, \frac{N}{N-2})$, there exists a universal constant C (depending only on p, P and N) such that the following is true. Given nonnegative functions $a_i = a_i(t,x)$, for $1 \le i \le P$, such that*

$$\rho = \sum_{i=1}^{P} a_i \in L^\infty(-3,0;\mathcal{L}_w(\mathbb{R}^N)), \qquad \nabla\sqrt{a_i} \in L^2((-3,0)\times\mathbb{R}^N), \ \ 1 \le i \le P,$$

we have

$$a_i \in L^q(-3,0;L^p(B(0,3))),$$

for q such that $\frac{1}{p} = 1 - \frac{2}{qN}$. Moreover,

$$\|a_i\|_{L^q(-3,0;L^p(B(0,3)))} \le C\|\nabla_x\sqrt{a}_i\|^{\frac{2(p-1)}{p}}_{L^2((-3,0)\times R^N)}\|\rho\|^{\frac{1}{p}}_{L^\infty(-3,0;\mathcal{L}_w(\mathbb{R}^N))}. \tag{26}$$

Proof. By Lemma 13, there exists a constant C such that the inequality (25) holds

$$||\rho(t,\cdot)||_{L^1(B(0,3))} \le C||\rho(t,\cdot)||_{\mathcal{L}_w(\mathbb{R}^N)} \ \text{ for } \ -3 < t < 0.$$

And so, using the nonnegativity of the functions, for any $1 \le i \le P$, we have

$$\|a_i\|_{L^\infty(-3,0;L^1(B(0,3)))} \le C||\rho||_{L^\infty(-3,0;\mathcal{L}_w(\mathbb{R}^N))}.$$

Moreover, by the Sobolev imbedding

$$\|a_i\|_{L^1(-3,0;L^{\frac{N}{N-2}}(R^N))} \le C\|\nabla_x\sqrt{a}_i\|_{L^2((-3,0)\times R^N)}.$$

Now, let $0 < \theta < 1$, and q defined by

$$\begin{cases} \dfrac{1}{p} = \dfrac{1-\theta}{1} \\ \dfrac{1}{q} = \dfrac{\theta}{1} + \dfrac{1-\theta}{\frac{N}{N-2}} \end{cases}$$

Then, by standard interpolation, it follows that each a_i belongs to the space $L^q(0,T_0;L^p(B(0,3)))$, with the bound

$$\|a_i\|_{L^q(-3,0;L^p(B(0,3)))} \le C(B(0,3))\|a_i\|^\theta_{L^1(-3,0;L^{\frac{N}{N-2}}(\mathbb{R}^N))}\|a_i\|^{1-\theta}_{L^\infty(-3,0;L^1(B(0,3)))}. \tag{27}$$

The inequality (26) follows readily by the definition of ρ. ∎

5 Scaling argument

This section is dedicated to the proof of Theorem 1.

The proof is done by a contradiction argument. First, by standard short time existence results, we know that there exists a solution for some short time. Next, we assume that the maximal time of existence, which we denote by T_0, is finite. We will show that the L^∞ norm of $a = (a_1, \cdots, a_P)$ is uniformly bounded on $(\frac{T_0}{2}, T_0) \times \mathbb{R}^N$. But this will contradict the fact that the solution blows up at T_0.

We introduce now the rescaled solutions. Let $a = (a_1, \cdots, a_P)$ be a solution of (1). Let $\frac{T_0}{2} < T < T_0$, $x_0 \in \mathbb{R}^N$ and $0 < \varepsilon < \sqrt{T_0/6}$. Then, we define the rescaled functions

$$a^\varepsilon(s,y) = \varepsilon^{\frac{2}{\nu-1}} a(\varepsilon^2 s + T, \varepsilon y + x_0), \; -\frac{T}{\varepsilon^2} \leq s \leq \frac{T_0 - T}{\varepsilon^2}, \; y \in \mathbb{R}^N. \tag{28}$$

The function $a^\varepsilon = (a_1^\varepsilon, \cdots, a_P^\varepsilon)$ is a solution to a system (1) with

$$Q^\varepsilon(a) = \varepsilon^{\frac{2\nu}{\nu-1}} Q\left(\varepsilon^{\frac{-2}{\nu-1}} a\right).$$

The reaction term $Q^\varepsilon(a)$ verifies Hypothesis (2), (3), (4), and (5) with the constant Λ as in (3) independent on ε. The $\mathcal{L}_w$ norm of the function ρ^ε rescales by the following identity

$$\|\rho^\varepsilon\|_{L^\infty(-\frac{T}{\varepsilon^2},0;\mathcal{L}_w(\mathbb{R}^N))} = \varepsilon^{\frac{2}{\nu-1}-2} \|\rho\|_{L^\infty(0,T;\mathcal{L}_w(\mathbb{R}^N))}.$$

This implies, by Corollary 12, that the following bound holds

$$\|\rho^\varepsilon\|_{L^\infty(-3,0;\mathcal{L}_w(\mathbb{R}^N))} \leq C\varepsilon^{\frac{2}{\nu-1}-2}(\|\rho^0\|_{L^1(\mathbb{R}^N)} + \|\rho^0\|_{L^\infty(\mathbb{R}^N)}). \tag{29}$$

Through the rescaling, from (29), we can control the following $L^{\bar{q}}$ norm.

Lemma 15 *There exists $\bar{q} > 1$ and $\varepsilon_0 > 0$ (depending on $\bar{d}$, $\underline{d}$, Λ, P, N, T_0 and M_0) such that for all $\varepsilon \leq \varepsilon_0$ we have*

$$\sum_{i=1}^P \|a_i^\varepsilon\|_{L^{\bar{q}}((-3,0)\times B(0,3))} \leq \delta^\star,$$

where $\delta^\star = \delta^\star(\bar{q})$ is the same as in Proposition 4.

Proof. Let $0 < \varepsilon_0 < \sqrt{T_0/6}$. We apply Corollary 14 to the rescaled solutions a^ε.

Each component $a_i^\varepsilon \in L^q(-3,0;L^p(B(0,3)))$ for $p \in (1,\frac{N}{N-2})$ and q such that $\frac{1}{p} = 1 - \frac{2}{qN}$; Moreover, for each $1 \leq i \leq P$

$$\|a_i^\varepsilon\|_{L^q(-3,0;L^p(B(0,3)))} \leq C\|\nabla_x \sqrt{a_i^\varepsilon}\|_{L^2((-3,0)\times R^N)}^{\frac{2(p-1)}{p}} \|\rho^\varepsilon\|_{L^\infty(-3,0;\mathcal{L}_w(\mathbb{R}^N))}^{\frac{1}{p}},$$

$$\|\nabla_x \sqrt{a_i^\varepsilon}\|_{L^2((-3,0)\times\mathbb{R}^N)}^2 \leq \varepsilon^{\frac{2}{\nu-1}-N-1}\|\nabla_x\sqrt{a_i}\|_{L^2((0,T_0)\times\mathbb{R}^N)}^2. \tag{30}$$

Hence, together with (29), we obtain:

$$\sum_{i=1}^{P} \|a_i^\varepsilon\|_{L^q(-3,0;L^p(B(0,3)))} \leq C(M_0,T_0)\varepsilon^{\alpha(p)}$$

where $\alpha(p) := \frac{p-1}{p}(\frac{2}{\nu-1} - N - 1) + \frac{1}{p}(\frac{2}{\nu-1} - 2)$ and $C(M_0,T_0)$ is a constant depending on the quantity M_0 defined by (7) and T_0.

Since $\nu < 2$, the factor $(\frac{2}{\nu-1} - 2)$ is positive. Thus, there exists $\bar{p} \in (1,\frac{N}{N-2})$ such that $\bar{\alpha} := \alpha(r) > 0$ and

$$\sum_{i=1}^{P} \|a_i^\varepsilon\|_{L^{\bar{q}}((-3,0)\times B(0,3))} \leq C(M_0,T_0)\varepsilon^{\bar{\alpha}}$$

with $r = \min(\bar{p},\bar{q})$, and $\bar{q} = \frac{2}{N}\left(\frac{\bar{p}}{\bar{p}-1}\right)$. This last inequality implies that it is enough to choose $\varepsilon_0 \leq \inf\left\{\sqrt{\frac{T_0}{6}}, \left(\frac{\delta^\star}{C(M_0,T_0)}\right)^{\frac{1}{\bar{\alpha}}}\right\}$ to conclude the statement for all $0 < \varepsilon \leq \varepsilon_0$.

∎

Next, we apply Proposition 4 to the rescaled functions a^ε.

Corollary 16 *Let $\delta^\star = \delta^\star(\bar{q})$, $\bar{\alpha}$ and $C(M_0,T_0)$ as in Lemma 15. Then, for all $\frac{T_0}{2} < T < T_0$, we have*

$$\sum_{i=1}^{P} ||a_i(T,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq \left(\inf\left\{\sqrt{\frac{T_0}{6}}, \left(\frac{\delta^\star}{C(M_0,T_0)}\right)^{\frac{1}{\bar{\alpha}}}\right\}\right)^{-\frac{2}{\nu-1}}.$$

Proof. Let x_0 be any point in $\mathbb{R}^N$, $\frac{T_0}{2} < T < T_0$. Let a^{ε_0} be the rescaled function defined in (28) with $\varepsilon_0 = \inf\left\{\sqrt{\frac{T_0}{6}}, \left(\frac{\delta^\star}{C(M_0,T_0)}\right)^{\frac{1}{\bar{\alpha}}}\right\}$. Then, by Lemma 15, we can apply Proposition 4 to a^{ε_0} to obtain:

$$0 \le \sum_{i=1}^{P} a_i(x_0, T) = \varepsilon_0^{\frac{-2}{\nu-1}} \sum_{i=1}^{P} a_i^{\varepsilon_0}(0,0) \le \varepsilon_0^{\frac{-2}{\nu-1}} P.$$

∎

Finally, we can now prove Theorem 1.

Assume by contradiction that the maximal time of existence of a solution $a = (a_1, \cdots, a_P)$, T_0 is finite. Then, by Corollary 16, each $a_i(t, \cdot)$, for $1 \le i \le P$ would be uniformly bounded for all $\frac{T_0}{2} < t < T_0$. In particular, it would be bounded at $t = T_0$. Hence, by standard arguments this would imply that $a \in C^\infty$ at $t = T_0$. This concludes the proof because it would negate the fact that T_0 is the maximal time of existence of a smooth solution $a = (a_1, \cdots, a_P)$.

A Proof of Theorem 2

This appendix is dedicated to the proof of the maximum principle in the case $P = 2$. This is a very standard proof. We include it here since we did not find the result in the literature. The system is equivalent to the pair of equations

$$\begin{cases} \partial_t a_1 - d_1 \Delta a_1 = Q(a), & t > 0,\ x \in \mathbb{R}^N, \\ \partial_t a_2 - d_2 \Delta a_2 = -Q(a), & t > 0,\ x \in \mathbb{R}^N, \\ a(0,x) = a^0(x), & x \in \mathbb{R}^N. \end{cases} \tag{31}$$

The main remark is that, under Hypothesis (5), we have

$$Q(a_1, a_2)(a_1 - a_2) \le 0, \qquad \text{for any } a_1, a_2 \ge 0. \tag{32}$$

Let (a_1, a_2) be a smooth solution (C^2) to (31) on $([0,T] \times \mathbb{R}^N)$, decaying to 0 when $|x| \to \infty$. For any $\varepsilon > 0$, we define

$$a_1^\varepsilon(t,x) = a_1(t,x) - \varepsilon t,$$
$$a_2^\varepsilon(t,x) = a_2(t,x) - \varepsilon t.$$

Assume that $\sup(a_1^\varepsilon, a_2^\varepsilon)$ attains a local maximum at a point $(t_\varepsilon, x_\varepsilon) \in (0,T] \times \mathbb{R}^N$. Let say that its value is $a_1^\varepsilon(t_\varepsilon, x_\varepsilon)$. (The proof is similar for the other case.) Especially, this is a local maximum for a_1^ε. Note that

$$a_1^\varepsilon - a_2^\varepsilon = a_1 - a_2.$$

Therefore, from (32), we have

$$Q(a)(t_\varepsilon, x_\varepsilon) \le 0.$$

From the first equation of (31) we have

$$(\partial_t a_1^\varepsilon - d_1 \Delta a_1^\varepsilon)(t_\varepsilon, x_\varepsilon) = Q(a)(t_\varepsilon, x_\varepsilon) - \varepsilon \le -\varepsilon < 0.$$

This contradicts the fact that a_1^ε attains a local maximum at $(t_\varepsilon, x_\varepsilon)$. Hence

$$\sup_{(t,x)\in(0,T]\times\mathbb{R}^N} (a_1^\varepsilon(t,x), a_2^\varepsilon(t,x)) \le \sup_{x\in\mathbb{R}^N} (a_1^\varepsilon(t=0,x), a_2^\varepsilon(t=0,x)) = \sup_{x\in\mathbb{R}^N} (a_1^0(x), a_2^0(x)).$$

Passing to the limit as $\varepsilon \to 0$ gives the result.

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