# HIGHLY DEGENERATE HARMONIC MEAN CURVATURE FLOW

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Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2006

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*In this thesis we consider a fully nonlinear parabolic evolution equation for a hypersurface in the Euclidean space. We study the existence of solutions to the Harmonic Mean Curvature Flow for weakly convex surfaces. We prove short time existence for weakly convex surfaces with flat sides as well as optimal regularity. We also show that the boundaries of the flat sides evolve by the Curve Shortening Flow. Our results follow by using the Inverse Function Theorem in suitable Banach Spaces scaled according to the degenerate problem. We establish sharp a-priori estimates for the linearized problem in these spaces.*

# Contents

# Acknowledgments

I am deeply thankful to my advisor, Professor Panagiota Daskalopoulos for all the patience, enthusiasm and constant encouragement in these years. Her love for mathematics and positive attitude have been very inspirational for me. She has continuously oriented me in the correct research direction. Without Toti this thesis could not have been completed.

I would like to express my profound gratitude to Professor Richard Hamilton. He has always been available to answer all my questions in many discussions we had. It has been an unforgettable experience to work close to him for the extraordinary approach he has towards mathematics.

Many thanks go to my friends Natasa and Sean that gave me valuable feedback and constructive criticism. I would like to thank my friend Henry for being always there in important moments of my time in New York City. I will always be indebted to Professors Robert Friedman and Enrico Jannelli without whom this experience could not have started.

I thank my family for teaching me respect and strength of will.

*To my parents and Nikhil*

# Chapter 1

# Introduction

In this thesis we study the evolution of a surface under the *Harmonic Mean Curvature Flow* (*HMCF*). This flow is one example of the numerous geometric flows which have been studied in recent years [17], [2], [3], [11], [19].

Some of these flows arise from the evolution of a hypersurface in $\mathbb{R}^n$ by specific functions of their principal curvatures. Common choices of these functions include the Gauss, mean, inverse mean, harmonic mean and inverse harmonic mean curvatures. In general, geometric flows evolve certain geometric quantities by diffusive type equations. Among these flows we find the porous medium equation, the p-Laplacian and the harmonic map. Other examples are the Ricci, the Calabi flows of manifolds and the Yamabe flows which are conformal flows of metrics.

The increased interest in geometric flows may, in part, be explained by their relevance to the study of problems in both mathematics and the physical sciences. These flows have been applied in a variety of settings. For instance, the mean curvature flow in minimal surface theory; the inverse mean curvature flow in the theory of relativity [21]; the Ricci flow in resolving the Poincaré Conjecture [27]; the Gauss curvature flow in the modeling of the wearing of stones [13]; and several other flows in applications such as image processing [26].

In this thesis we analyze the evolution of some particular surfaces with flat sides under the *HMCF*. For such surfaces the results established in this thesis distinguish the *HMCF* from other related flows (such as the Gauss curvature flow with flat sides, the porous medium equation and the p-Laplacian equation) in the following principal respects:

1. A surface evolving by the *HMCF* remains of the same class.
2. The motion of the free boundary of a surface with flat side moves by the Curve Shortening Flow. This has the consequence that the free boundary is specified for all positive times.

These two features make the *HMCF* particularly interesting. At the time of writing, no other flow is known to have a similar behavior. The fact that the solution $\Sigma(t)$ remains in the class $C^{1,\alpha}$, for $t > 0$, distinguishes this flow from other, previously studied, degenerate free-boundary problems for which the regularity of the solution up for $t > 0$ does not depend on the regularity of the initial data.

In the following sections we introduce the *HMCF* for a strictly convex surface and then we derive it for a weakly convex surface. We also give some examples of how the flow behaves for simple initial surfaces. We discuss the known results due to Andrews [1] and Diëter [12]. The chapter concludes by stating the main results of this thesis.

## 1.1 The flow

We study the evolution of a weakly convex surface $\Sigma$ in the space $\mathbb{R}^3$ under the *Harmonic Mean Curvature Flow* (*HMCF*):

$$\frac{\partial P}{\partial t} = \frac{K}{H}\ \vec{N} \tag{1.1}$$

where each point $P$ moves in the inward normal direction $\vec{N}$ with velocity given by the ratio between the *Gauss curvature* $K$ and the *Mean curvature* $H$.

If we denote by $\lambda_1$ and $\lambda_2$ the two principal curvatures of a strictly convex surface, then we can express $\frac{K}{H}$ as:

$$\frac{K}{H} = \frac{\lambda_1\,\lambda_2}{\lambda_1 + \lambda_2}$$

which is the *harmonic mean* of the principal curvatures $\lambda_1$, $\lambda_2$.

## 1.2 Examples

With the following examples we aim to provide a first introduction to the *HMCF* for particular initial surfaces. In the first example we have a strictly convex surface, in the second a weakly convex surface which is a surface of revolution.

**Example 1.** *We observe that* $\dfrac{K}{H} = H$ *iff* $\lambda_1 = \lambda_2$*. Thus the HMCF is the same as the mean curvature flow for a sphere. In particular, if* $\Sigma$ *is a sphere of radius* $1$ *centered at the origin and we flow it by the HMCF, then its radius* $r(t)$ *evolves by:*

$$\frac{d\,r}{d\,t} = -\frac{1}{r(t)}$$

*Hence, the sphere shrinks by the equation* $r(t) = \sqrt{1 - 2\,t}$ *and it disappears at time* $t = 1/2$*.*

**Example 2.** *We consider a surface of revolution* $\Sigma_t$*. Suppose that* $\Sigma_t$ *is given by rotating the graph of a function* $f(\cdot, t)$ *about the* $x$*-axis where* $f(x,t) = g(x) + h(t)$*, for instance,* $f(x,t) = -cosh(x) + \sqrt{a - 2\,t}$*, on the interval* $[-1, 1]$ *with* $a = cosh(1)$*. Then, if* $\Sigma_t$ *evolves by the HMCF, we have that the function* $f$ *solves the parabolic equation:*

$$f_t = \frac{f_{xx}}{1 + f_x^2 - f\,f_{xx}}$$

*If we assume that* $g$ *is a solution to a minimal surface of revolution,* $1 + g_x^2 - g\,g_{xx} = 0$*, then* $f$ *solves* $f_t = \dfrac{f_{xx}}{1 + f_x^2 - f f_{xx}}$ *if we choose* $h'(t) = -\dfrac{1}{h(t)}$*.*

*Remark* 1. *Note that the free boundary of the surface of revolution given by the function $f$ in the example 2 which is of class $C^{0,\alpha}$ does not move by the Curve Shortening Flow.*

## 1.3 Prior Results

The existence of solutions to the *HMCF* with strictly convex, smooth initial data was first shown by Andrews in [2]. He proved that under the *HMCF*, strictly convex, smooth surfaces converge to round points in finite time. Later, Diëter established in [12] the short time existence for a class of flows, including the *HMCF*, with weakly convex, smooth initial data. More precisely, she showed that if at time $t = 0$ a surface $\Sigma$ satisfies $K \geq 0$ and $H > 0$, then there exists a unique strictly convex, smooth solution $\Sigma_t$, for $0 < t < \tau$ for some $\tau > 0$. This solution will exist up to the time where its enclosed volume becomes zero. However, Diëter did not address the highly degenerate case where the initial data is weakly convex and both $K$ and $H$ may vanish at a region.

## 1.4 New Results

The main result of this work concerns the existence and regularity of the solutions of the *HMCF* for weakly convex surfaces with a flat side. We study an initial surface $\Sigma$ of class $C^{1,\gamma}$, $0 < \gamma \leq 1$, which is given by the union of two surfaces $\Sigma = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1$ is flat and $\Sigma_2$ is smooth and strictly convex. We assume that the lower part of the surface $\Sigma$ can be written as the graph of a function $h$. We define $g := h^p$, for some $0 < p < 1$. Our main assumption on the surface $\Sigma$ is that it satisfies the following non-degeneracy condition:

$$|Dg| \geq \lambda \qquad \text{and} \qquad G = (g_{ij}) = \begin{pmatrix} (q-1)\, g_\nu^2 + g\, g_{\nu\nu} & g_{\nu\tau} \\ g_{\nu\tau} & g_{\tau\tau} \end{pmatrix} \geq \lambda \qquad (I)$$

for some numbers $\lambda > 0$, $q = 1/p$, where $\nu$ and $\tau$ denote, respectively, the normal and tangential directions to the level sets of $g$; $g_{\nu\tau}$, $g_{\nu\nu}$ and $g_{\tau\tau}$ denote the second order derivatives in these directions.

Based on the above conditions, our main results show that:

1. The *HMCF* admits a viscosity solution $\Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$ of class $C^{1,\gamma}$ which is smooth up to the *interface* $\Gamma_t = (\Sigma_1)_t \cap (\Sigma_2)_t$.

2. The flat side $(\Sigma_1)_t$ persists for some positive time and, in particular, $\Gamma_t$ evolves by the *Curve Shortening Flow.*

The fact that the solution $\Sigma_t$ remains in the class $C^{1,\gamma}$, for $t > 0$, distinguishes this flow from other, previously studied, degenerate free-boundary problems (such as the Gauss curvature flow with flat sides, the porous medium equation and the p-Laplacian equation) in which the regularity of the solution up for $t > 0$ does not depend on the regularity of the initial data.

## 1.5 Outline

The rest of the thesis is organized as follows: in Chapter 2, we derive the partial differential equation that will be used throughout. We also introduce the notation and we give the definitions that will be used for proving the main result. In Chapter 3 we state the main theorem of the thesis and derive the fundamental estimates to prove the main result by doing a local change of coordinates which gives the key ingredient for the proof of the existence of a solution to the *HMCF*. In Chapter 4, we study the regularity question. We show the main results of the thesis, the existence and smoothness of the solution. We also prove that the free boundary moves by the Curve Shortening Flow. To conclude, in Chapter 5, we provide an interpretation of our solutions showing that they are viscosity solutions. We conclude the thesis by showing an interesting geometric property of the solutions to the *HMCF*.

# Chapter 2

# Preliminaries

## 2.1 Notation

Throughout the thesis we will use the following notation. Let $\mathcal{A}$ be a compact subset of the half space $\{\,(x,y) : \ x \geq 0\,\}$ such that $(0,0) \in \mathcal{A}$. Then, we define:

$$\begin{aligned}
\mathcal{A}^{\circ} &:= \{\ y \in \mathbb{R} : (0,y) \in \mathcal{A}\,\} \\
\tilde{\mathcal{A}} &:= \{(z,y) \in \mathbb{R}^2 : z = ln(x),\ (x,y) \in \mathcal{A},\ x \neq 0\} \\
Q_T &:= \mathcal{A} \times [0,T],\ T > 0 \\
Q_T^{\circ} &:= \mathcal{A}^{\circ} \times [0,T] \\
\tilde{Q}_T &:= \tilde{\mathcal{A}} \times [0,T]
\end{aligned}$$

Let $0 < p < 1$. Given a function $f$ defined on $\mathcal{A}$ we will denote:

$$\begin{aligned} f^{\circ}(y) &:= f(0,y) \\ \tilde{f}(z,y) &:= e^{-p\,z}\,(\,f(x,y) - f^{\circ}(y)\,) \end{aligned}$$

with $z = ln(x)$, for $x > 0$

Analogously, given a function $f$ defined on $Q_T$ we will denote by

$$\begin{aligned} f^{\circ}(y,t) &:= f(0,y,t) \\ \tilde{f}(x,y,t) &:= e^{-p\,z}\,(\,f(x,y,t) - f^{\circ}(y,t)) \end{aligned}$$

*Remark* 2. *Note that we abuse notation and omit the dependency on p when denoting $\tilde{f}$.*

## 2.2 The Banach Space $C_s^{2+\alpha,p}$

The goal of this section is to define the space $C_s^{2+\alpha,p}$. We begin by defining the hyperbolic metric $s$. We introduce the natural parabolic extension of $s$, $\tilde{s}$. We define the Hölder spaces scaled first according to $s$, and then to $\tilde{s}$.

**Definition** 1. *Given a subspace $\mathcal{A}$ as above, we define the hyperbolic distance $s(P_1,P_2)$ such that for $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$ with*

$$\{P_1,P_2\} \subseteq \mathcal{A} \cap \{(x,y) \in \mathbb{R}^2 : x > 0\}$$

*we have :*

$$s(P_1,P_2) := \sqrt{|ln(x_1) - ln(x_2)|^2 + |y_1 - y_2|^2} \qquad if\ 0 < x_1,\, x_2 \le 1$$

*and it is equivalent to the standard euclidean metric otherwise.*

**Definition** 2. *Given* $T > 0$, $Q_T = \mathcal{A} \times [0,T]$, *we define the hyperbolic distance on* $Q_T$, $\tilde{s}(\tilde{P}_1, \tilde{P}_2)$ *such that for* $\tilde{P}_1 = (x_1, y_1, t_1)$, $\tilde{P}_2 = (x_2, y_2, t_2)$ *with*

$$\{\tilde{P}_1, \tilde{P}_2\} \subseteq Q_T \cap \{(x,y,t) \in \mathbb{R}^2 \times [0,T] : x > 0\}$$

*we have*

$$\tilde{s}(\tilde{P}_1, \tilde{P}_2) := s(P_1, P_2) + \sqrt{|t_1 - t_2|} \text{ if } 0 < x_1, x_2 \leq 1$$

$$\text{where} \quad P_1 = (x_1, y_1),\ P_2 = (x_2, y_2) \quad .$$

Let $0 < \alpha \leq 1$. We can define Hölder continuity in terms of the hyperbolic distance. We say that a continuous function $f$ on a compact subset $\mathcal{A}$ is Hölder continuous with respect to the metric $s$ if there exists $C > 0$ such that for all points $P_1$ an $P_2$ in $\mathcal{A}$ we have:

$$| f(P_1) - f(P_2) | \leq C\, s[P_1, P_2]^{\alpha}.$$

We define the Hölder semi-norm:

$$\| f \|_{H_s^{\alpha}(\mathcal{A})} := \sup_{P_1 \neq P_2 \in \mathcal{A} \cap \{(x,y) \in \mathbb{R}^2 : x > 0\}} \frac{| f(P_1) - f(P_2) |}{s[P_1, P_2]^{\alpha}}$$

We also define the norms:

$$\begin{aligned} \| f \|_{C_s^{\alpha}(\mathcal{A})} &:= \| f \|_{C^0(\mathcal{A})} + \| f \|_{H_s^{\alpha}(\mathcal{A})} \\ || f ||_{C^0(\mathcal{A})} &:= \sup_{P \in \mathcal{A}} | f(P) | \end{aligned}$$

**Definition** 3. *We say that a function* $f$ *belongs to* $C^{\alpha,p}_s(\mathcal{A})$ *if all of the following hold:*

*i.* $f^\circ \in C^\alpha(\mathcal{A}^\circ)$.

*ii.* $\tilde{f} \in C^\alpha(\tilde{\mathcal{A}})$.

*The norm of* $f$ *in the space* $C^{\alpha,p}_s(\mathcal{A})$ *is defined as:*

$$||\, f\, ||_{C^{\alpha,p}_s(\mathcal{A})} := ||\, f^\circ\, ||_{C^\alpha(\mathcal{A}^\circ)} + ||\, \tilde{f}\, ||_{C^\alpha(\tilde{\mathcal{A}})}$$

*We also define the norm* $||\, f\, ||_{C^{0,p}(\mathcal{A})} := ||\, f^\circ\, ||_{C^0(\mathcal{A}^\circ)} + ||\, \tilde{f}\, ||_{C^0(\tilde{\mathcal{A}})}$.

*Remark* 3. *Note that* $g(z,y) \in C^\alpha(\tilde{\mathcal{A}})$ *iff* $g(x,y) \in C^\alpha_s(\mathcal{A})$, *where* $z = ln(x)$. *The hyperbolic metric is weaker than the Euclidean metric, hence, the space* $C^\alpha(\mathcal{A})$ *of the Hölder functions with respect to the Euclidean metric, is a subspace of* $C^\alpha_s(\mathcal{A})$. *Also, smooth functions belong to* $C^{\alpha,p}_s(\mathcal{A})$.

**Definition** 4. *We say that a continuous function* $f$ *on* $\mathcal{A}$ *belongs to* $C^{2+p}(\mathcal{A})$ *if all of the following hold:*

*i.* $f^\circ \in C^2(\mathcal{A}^\circ)$.

*ii.* $f$ *has continuous derivatives* $f_x$, $f_y$, $f_{xx}$, $f_{xy}$, *and* $f_{yy}$ *in the interior of* $\mathcal{A}$.

*iii.* $x^{-p}\,(f - f^\circ)$, $x^{1-p}\, f_x$, $x^{-p}\,(\, f_y - f^\circ_y\,)$, $x^{2-p}\, f_{xx}$, $x^{1-p}\, f_{xy}$ *and* $x^{-p}\,(\, f_{yy} - f^\circ_{yy}\,)$ *extend continuously up to the boundary.*

*The norm of* $f$ *in the space* $C^{2+p}(\mathcal{A})$ *is defined as follows:*

$$\|\, f \,\|_{C^{2+p}(\mathcal{A})} := \|\sum_{m=0}^{2} D_y^m\, f^\circ \,\|_{C^0(\mathcal{A}^\circ)} + \sum_{m+n=0}^{2} \|\, D_z^m\, D_y^n\, \tilde{f}\,\|_{C^0(\tilde{\mathcal{A}})}$$

**Definition** 5. *Given* $f \in C^{2+p}(\mathcal{A})$*, we say that* $f$ *belongs to* $C_s^{2+\alpha,p}(\mathcal{A})$ *if all of the following hold:*

*i.* $f^\circ \in C^{2+\alpha}(\mathcal{A}^\circ)$.

*ii.* $x\, f_x, f_y,\, x^2\, f_{xx}, x\, f_{xy}$ *and* $f_{yy}$ *extend continuously up to the boundary.*

*iii. The extensions are Hölder continuous on* $\mathcal{A}$ *of class* $C_s^{\alpha,p}(\mathcal{A})$.

*The norm of* $f$ *in the space* $C_s^{2+\alpha,p}(\mathcal{A})$ *is defined as:*

$$\|\, f \,\|_{C_s^{2+\alpha,p}(\mathcal{A})} := \|\, f^\circ \,\|_{C^{2+\alpha}(\mathcal{A}^\circ)} + \sum_{m+n=0}^{2} \|\, x^m\, D_x^m\, D_y^n\, f \,\|_{C_s^{\alpha,p}(\mathcal{A})}$$

*Remark* 4. *By definition:* $\tilde{f}_z = -p\,\tilde{f} + x^{1-p}\, f_x, \quad \tilde{f}_{zz} = -p\,\tilde{f}_z + (1-p)\, x^{2-p}\, f_{xx}$. *Hence:*

$$\sum_{m+n=0}^{2} \|\, x^m\, D_x^m\, D_y^n\, f \,\|_{C_s^{\alpha,p}(\mathcal{A})} \simeq \|\, \tilde{f}\,\|_{C^{2+\alpha}(\tilde{\mathcal{A}})}$$

*Remark* 5. $f \in C_s^{2+\alpha,p}(\mathcal{A})$ *iff* $f^\circ \in C^{2+\alpha}(\mathcal{A}^\circ)$ *and* $\tilde{f} \in C^{2+\alpha}(\tilde{\mathcal{A}})$*. Hence, the norm*

$$\|\, f \,\|_{C_s^{2+\alpha,p}(\mathcal{A})} \simeq \|\, f^\circ \,\|_{C^{2+\alpha}(\mathcal{A}^\circ)} + \|\, \tilde{f}\,\|_{C^{2+\alpha}(\tilde{\mathcal{A}})}$$

Let $T > 0$. The definitions above can be naturally extended on the space-time domain $Q_T$ by using the parabolic distance $d\bar{s}^2 = ds^2 + |dt|$.

**Definition** 6. *We define the space $C_s^\alpha(Q_T)$ to be the standard Hölder space with respect to the metric $d\bar{s}^2$.*

**Definition** 7. *We say that a continuous function $f$ on $Q_T$ belongs to $C^{2+p}(Q_T)$ if all of the following hold:*

*i. $f$ has continuous derivatives $f_t, f_x, f_y, f_{xx}, f_{xy}$ and $f_{yy}$ in the interior of $Q_T$.*

*ii. $f^\circ$ has continuous derivatives that extend continuously up to the boundary.*

*iii. $x^{-p}\,(f-f^\circ),\ x^{-p}\,(f_t-f_t^\circ),\ x^{1-p}f_x, x^{-p}\,f_y,\ x^{2-p}f_{xx},\ x^{1-p}f_{xy}$ and $x^{-p}\,(f_{yy}-f_{yy}^\circ)$ extend continuously up to the boundary.*

*The norm of $f$ in the space $C^{2+p}(Q_T)$ is defined as follows:*

$$||\,f\,||_{C^{2+p}} := ||\,f^\circ\,||_{C^2} + \sum_{l+m+2j=0}^{2} ||\,D_z^l\,D_y^m\,D_t^j\,\tilde{f}||_{C^\circ}$$

**Definition** 8. *The function $f$ belongs to $C_s^{2+\alpha,p}(Q_T)$ if all of the following hold:*

*i. $f \in C^{2+p}(Q_T)$.*

*ii. $f, f_t, xf_x, f_y, x^2 f_{xx}, xf_{xy}$ and $f_{yy}$ belong to $C_s^{\alpha,p}(Q_T)$.*

Let $k$ be a positive integer. We can extend these definitions to spaces of higher order derivatives.

**Definition** 9. *We denote by* $C^{k,p}(Q_T)$ *the space of all functions* $f$ *whose* $k$*-th order derivatives* $D_x^i\, D_y^j\, D_t^l\, f,\ i+j+2l=k$ *in the interior of* $Q_T$ *and* $x^i\, D_x^i\, D_y^j D_t^l\,(f-f^\circ\,),\ i+j+2l=k$ *exist and belong to the space* $C^0(Q_T)$. *We define* $C^{\infty,p}(Q_T)=\cap_k\, C^{k,p}(Q_T)$.

**Definition** 10. *We denote by* $C_s^{k+\alpha,p}(Q_T)$ *the space of all functions* $f\in C^{k,p}(Q_T)$ *such that* $x^i\, D_x^i\, D_y^j\, D_t^l\, f,\ i+j+2l=k$ *belong to the space* $C_s^{\alpha,p}(Q_T)$. *This space* $C_s^{k+\alpha,p}(Q_T)$ *is equipped with the norm:*

$$||\, f\,||_{C_s^{k+\alpha,p}(Q_T)} := \sum_{i+j+2l\le k} ||\, x^i\, D_x^i D_y^j D_t^l\, f\,||_{C_s^{\alpha,p}(Q_T)}$$

*Remark* 6. *A function* $f\ \in\ C_s^{k+\alpha,p}(Q_T)$ *iff* $f^\circ\ \in\ C^{k+\alpha}(Q_T^\circ)$ *and* $\tilde f\ \in\ C^{k+\alpha}(\tilde Q_T)$. *Moreover,*

$$||\, f\,||_{C_s^{k+\alpha,p}(Q_T)} \simeq ||\, f^\circ\,||_{C^{k+\alpha,[k/2]+\alpha/2}(Q_T^\circ)} + ||\, \tilde f\,||_{C^{k+\alpha,[k/2]+\alpha/2}(\tilde Q_T)}$$

**Proposition 11.** *The space* $C_s^{k+\alpha,p}(Q_T)$ *is a Banach space for every positive integer* $k$.

*Proof.* Let $\{f_n\}$ be a Cauchy sequence of functions in $C_s^{k+\alpha,p}(Q_T)$. Then, the sequences $f_n^\circ$ and $\tilde f_n$ defined as in Section 2.2, are Cauchy sequences in their respective spaces $C^{k+\alpha}(Q_T^\circ)$ and $C^{k+\alpha}(\tilde Q_T)$. These spaces are Banach spaces, hence, there exist $f^\circ\ \in\ C^{k+\alpha,[k/2]+\alpha/2}(Q_T^\circ)$ and $\tilde f\ \in C^{k+\alpha,[k/2]+\alpha/2}(\tilde Q_T)$ such that

$$\lim_{n\to\infty} f_n^\circ = f^\circ \qquad in \qquad C^{k+\alpha,[k/2]+\alpha/2}(Q_T^\circ)$$

$$\lim_{n\to\infty} \tilde{f}_n = \tilde{f} \qquad in \qquad C^{k+\alpha,[k/2]+\alpha/2}(\tilde{Q}_T)$$

Hence, the function $f : Q_T \longmapsto \mathbb{R}$ defined such that for every $(x, y, t) \in Q_T$:

$$f(x, y, t) = f^\circ(y, t) + x^p\, \tilde{f}(ln(x), y, t)$$

belongs to $C_s^{k+\alpha,p}(Q_T)$ and by remark (6):

$$\lim_{n\to\infty} f_n = f \qquad in \qquad C_s^{k+\alpha,p}(Q_T).$$

□

## 2.3 Deriving the PDE

Let $\Sigma$ be a compact, smooth, strictly convex surface embedded in $\mathbb{R}^3$ which evolves by the *HMCF*. Assume that locally it can be seen as the graph of a function $f$. Let $P$ be a point of $\Sigma$, let $T_P\Sigma$ be the tangent space of $\Sigma$ at $P$. Let $\vec{V} \in T_P\Sigma$ and $p = p(t)$ a path on $\Sigma$ s.t.

$$\left.\frac{dp}{dt}\right|_{t=0} = \vec{V}$$

We derive the partial differential equation satisfied by the function $f$ by computing the Gauss curvature $K$ and the mean curvature $H$ in terms of the function $f$.

Then,

$$\vec{A}=\frac{d\vec{V}}{dt}=\frac{d^2p}{dt^2} \qquad \text{and} \qquad \vec{A}=\vec{A}_T+\vec{A}_N$$

where $\vec{A}_T$ and $\vec{A}_N$ are, respectively, the tangential and normal components of the acceleration vector $\vec{A}$.

The second fundamental form:

$$\Pi : T_P\Sigma \times T_P\Sigma \to T_P\Sigma$$

is a bilinear operator acting on the tangent space of $\Sigma$ defined on the diagonal:

$$\Pi(\vec{V},\vec{V}) = \vec{A}_N$$

for any vector $\vec{V}$. Since $\Pi$ is symmetric, it is enough to define it on the diagonal.

*Remark* 7. $\Pi$ *is also independent of the path.*

**Example 3.** *On the unit sphere:*

$$\Pi(\vec{V},\vec{V}) = |\vec{V}|^2 \quad \vec{N}$$

*where* $\vec{N}$ *is the unit normal vector.*

Assume that $\Sigma$ is the graph of a function $f$, i.e. $z = f(x,y)$. We compute the tangent vectors $\vec{V}$ and $\vec{W}$ in terms of $f$:

$$\vec{V}= \frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial z}, \qquad \vec{W}= \frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z}$$

Since $f(x,y) - z = 0$ on the graph, we have

$$\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial z}(f(x,y) - z) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial x} = 0$$

$$\vec{V}= \begin{pmatrix} 1 \\ 0 \\ f_x(x,y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}, \qquad \vec{W}= \begin{pmatrix} 1 \\ 0 \\ f_y(x,y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ f_y \end{pmatrix}$$

Next we compute the acceleration vector $\vec{A}$ and the normal vector $\vec{N}$:

$$\vec{A}= \frac{d\,\vec{V}}{dt} = \begin{pmatrix} 0 \\ 0 \\ f_{xx} \end{pmatrix} \qquad \vec{V} \times \vec{W}= \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$$

$$\vec{N}= \frac{\vec{V} \times \vec{W}}{\mid \vec{V} \times \vec{W} \mid} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$$

Hence $\mid \vec{A_N} \mid = \vec{A} \cdot \vec{N} = \dfrac{f_{xx}}{\sqrt{1+f_x^2+f_y^2}}$. By definition of the second fundamental form, we obtain:

$$\Pi(\vec{V},\vec{V}) = \frac{f_{xx}}{\sqrt{1+f_x^2+f_y^2}}\ \vec{N}$$

Similarly we can compute that:

$$\Pi(\vec{W},\vec{W}) = \frac{f_{yy}}{\sqrt{1+f_x^2+f_y^2}}\ \vec{N} \qquad \Pi(\vec{V},\vec{W}) = \frac{f_{xy}}{\sqrt{1+f_x^2+f_y^2}}\ \vec{N}$$

Thus, the second fundamental form $\Pi$ is given by the following expression:

$$\Pi = \frac{1}{\sqrt{1+f_x^2+f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \cdot \vec{N}$$

In order to compute the Gauss curvature $K$ and mean curvature $H$ in terms of $f$ we examine the *metric* $g_{ij}$ induced by $f$:

$$g_{xx} = g(\vec{V},\vec{V}) = 1+f_x^2, \quad g_{xy} = g(\vec{V},\vec{W}) = f_x\, f_y, \quad g_{yy} = g(\vec{W},\vec{W}) = 1+f_y^2$$

In order of computing the mean curvature $H$ and the Gauss curvature $K$ (which are respectively, the trace and the determinant of the second fundamental form with respect to the metric $\{g_{i,j}\}$): we write the inverse matrix of $\{g_{ij}\}$:

$$\{g^{ij}\} = \frac{1}{1+f_x^2+f_y^2}\begin{pmatrix} 1+f_y^2 & -f_x\,f_y \\ -f_x\,f_y & 1+f_x^2 \end{pmatrix}$$

Hence, $H$ and $K$ can be evaluated in terms of $f$;

$$\begin{aligned} H: &= \text{trace}_g \Pi = g^{ij}\cdot \Pi_{ij} = g^{xx}\Pi_{xx} + 2\,g^{xy}\Pi_{xy} + g^{yy}\,\Pi_{yy} = \\ &= \frac{1}{(1+f_x^2+f_y^2)^{3/2}}\left(\,(1+f_y^2)\,f_{xx} - 2\,f_x\,f_y\,f_{xy} + (1+f_x^2)f_{yy}\,\right) \end{aligned}$$

$$K: = \det{}_g\,\Pi = \frac{\det \Pi}{\det g} = \frac{f_{xx}\,f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}$$

Therefore, if the surface $\Sigma$ evolves by the *HMCF*, then, the function $f$ evolves by the following partial differential equation:

$$\frac{\partial f}{\partial t} = \frac{f_{xx}\,f_{yy} - f_{xy}^2}{(1+f_y^2)\,f_{xx} - 2\,f_x\,f_y\,f_{xy} + (1+f_x^2)\,f_{yy}} \tag{2.1}$$

# Chapter 3

# Short Time Existence

We define the class of hypersurfaces $\mathfrak{S}$ to be the class of weakly convex, compact hypersurfaces $\Sigma$ in $\mathbb{R}^3$ so that $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ is a flat surface and $\Sigma_2$ is a strictly convex and smooth surface. Next, we give a definition of a solution to the *HMCF* for a given initial weakly convex surface.

**Definition** 12. *Given a surface $\Sigma \in \mathfrak{S}$, we say that a family of weakly convex surfaces $\Sigma_t$, with $0 < t \leq T$, $T > 0$, is a solution to the HMCF with initial surface $\Sigma$ if:*

*i.* $\dfrac{\partial P}{\partial t} = \dfrac{K}{H} \overrightarrow{N}$

*ii.* $dist(\Sigma_t, \Sigma) \rightarrow 0$ *as* $t \rightarrow 0$

for any point $P \in \Sigma_t$ such that the *Mean Curvature* $H > 0$ where $\overrightarrow{N}$, $K$ and $H$ are respectively, the outward normal, the Gauss curvature and the Mean curvature of $\Sigma_t$ at the point $P$.

## 3.1 Main Theorem

### Non-degeneracy Condition I

In this section we introduce the non-degeneracy condition I for the class $\mathfrak{S}$ of weakly convex, compact hypersurfaces $\Sigma$ in $\mathbb{R}^3$, $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ is a flat surface and $\Sigma_2$ is a strictly convex and smooth surface. We can assume that the lower part of $\Sigma_2$ can be written as the graph of a function $h$ over a domain $\Omega$ containing the flat side. We define $g := h^p$. We say that $\Sigma$ satisfies the *non-degeneracy condition I* if:

$$|Dg| \geq \lambda \qquad \text{and} \qquad G = (g_{ij}) = \begin{pmatrix} (q-1)\, g_\nu^2 + g\, g_{\nu\nu} & g_{\nu\tau} \\ g_{\nu\tau} & g_{\tau\tau} \end{pmatrix} \geq \lambda \qquad (I)$$

for some numbers $\lambda > 0$, $q = 1/p$, where $\nu$ and $\tau$ denote, respectively, the normal and tangential directions to the level sets of $g$; $g_{\nu\tau}$, $g_{\nu\nu}$ and $g_{\tau\tau}$ denote the second order derivatives in these directions.

**Theorem 13. (Main Theorem)** *Assume that at time $t = 0$, $\Sigma$ is a surface of class $\mathfrak{S}$ as above. Assume that $g$ is smooth up to the interface $\Gamma$ and it satisfies the condition I. Then, there exists a time $T > 0$ such that the HMCF admits a solution $\Sigma_t \in \mathfrak{S}$ on $0 \leq t \leq T$. Moreover, the function $g(\cdot, t) = h^p(\cdot, t)$ is smooth up to the interface $z = 0$ and satisfies condition I. In particular, the interface $\Gamma_t$ between the flat side and the convex side is a smooth curve for all $t$ in $0 < t \leq T$ and it moves by the Curve Shortening Flow.*

*Sketch of the proof.*

The *HMCF* can be seen as a free boundary problem arising from the degeneracy near the flat side of the fully nonlinear parabolic PDE which describes the flow. By introducing an appropriate global change of variables one can transform the free boundary problem in an *initial value problem* of the form

$$\begin{cases} Mw = 0 & \textit{on } \mathcal{D} \times [0,T] \\ w = w_0 & \textit{at} \quad t = 0 \end{cases}$$

on the cylinder $\mathcal{D} \times [0,T]$, where $\mathcal{D} = \{(x,y); x^2 + y^2 \leq 1\}$ and

$$Mw = w_t - F(t,u,v,w,Dw,D^2w)$$

is a fully non-linear operator which becomes degenerate at $\partial\mathcal{D}$. To show that the last problem admits a solution, one applies the Inverse Function Theorem between appropriately defined Banach spaces.

The linearization of the operator $M$ at a point $\bar{w}$ close to the initial data $w_0$ can be modeled, after we straighten the boundary, on the degenerate equation

$$f_t = x^2 a_{11} f_{xx} + 2\,x\,a_{12} f_{xy} + a_{22} f_{yy} + b_1 x\, f_x + b_2 f_y \tag{3.1}$$

on $x > 0$, and no extra conditions on $f$ along the boundary $x = 0$.

The diffusion in the above equation is governed by the hyperbolic metric

$d\bar{s}^2 = d\,s^2 + |dt|$ where:

$$d\,s^2 = \frac{dx^2}{x^2} + dy^2.$$

Notice that the distance (with respect to the metric $s$) of an interior point ($x > 0$) from the boundary ($x = 0$) is *infinite.* This distinguishes our problem from other, previously studied, degenerate free-boundary problems such as the degenerate Gauss curvature flow and the porous medium equation.

*Remark* 8. *We will actually show in next section that Theorem 13 holds under the weaker condition that $g$ belongs to $C_s^{2+\alpha}$ where the definition of $C_s^{2+\alpha}$ will be given then.*

*Remark* 9. *We would like to remark that the Main Theorem implies that an initial surface of class $C^{1+\gamma}$ evolving by the HMCF remains of class $C^{1+\gamma}$.*

It turns out that the model equation can be obtained by doing a local change of coordinates as it will be shown in the next sections. The plan is, therefore, to study the problem at first locally, then globally after we analyze the model equation in our scaled Banach spaces.

## 3.2 Local Change of Coordinates

Assume that the surface $\Sigma$ belongs to the class $\mathfrak{S}$ such that its flat part $\Sigma_1$ lies on the $z = 0$ plane and the strictly convex part $\Sigma_2$ belongs to the $z > 0$ half-space. Assume that $\Sigma_t$ is a solution to the *HMCF* on $[0, T]$, for some

$T > 0$. Let $P_0(x_0, y_0, 0)$ be a point on $\Gamma_{t_0}$ for $t_0 > 0$ sufficiently small. Then, the strictly convex part of surface $(\Sigma_2)_{t_0}$ can be expressed locally around $P_0$ as the graph of a function $z = h(x, y, t)$. Let $g$ be defined by $g = h^p$, for $0 < p < 1$, such that it satisfies condition I, i.e.:

$$|Dg| \geq \lambda \qquad \text{and} \qquad G = (g_{ij}) = \begin{pmatrix} (q-1)\, g_\nu^2 + g\, g_{\nu\nu} & g_{\nu\tau} \\ g_{\nu\tau} & g_{\tau\tau} \end{pmatrix} \geq \lambda$$

where $\lambda > 0$, $q = 1/p$. Without any loss of generality we can assume that $x_0 > 0$ and that the normal vector to $\Gamma_{t_0}$ facing outwards the flat side of $(\Sigma_1)_{t_0}$ is parallel to the $x$-axis. In this way at $P_0$ we have:

$$g_x(P_0) > 0 \qquad and \qquad g_y(P_0) = 0$$

By the implicit function theorem we can solve the equation $z = g(x, y, t)$ with respect to $x$ around $P_0$. We obtain the map

$$x = l(z, y, t)$$

defined all $(z, y, t) \in \mathcal{B}$ sufficiently close to $(0, y_0, t_0)$. We then define $f$ on $\mathcal{B}_p$ such that

$$f(z, y, t) = l(z^p, y, t)$$

for every $(z,y,t) \in \mathcal{B}_p$, where $(z,y,t) \in \mathcal{B}_p$ iff $(z^p,y,t) \in \mathcal{B}$. Hence:

$$\begin{aligned} x &= l(g(x,y,t),y,t) = l(h(x,y,t)^p,y,t) = f(h(x,y,t),y,t) \\ \text{and} &\quad h(f(z,y,t),y,t) = (g(f(z^p,y,t),y,t))^{1/p} = z \end{aligned}$$

We can repeat the same argument for every point $P_0 \in \Gamma_{t_0}$. Hence, there exist $P_l \in \Gamma_{t_0}$ and $\mathcal{B}_{P_l}$, $l \in I$, with $I$ finite such that the lower part of the surface $(\Sigma_2)_{t_0}$ can be written as the graph of $h$ over the domain $(\Sigma_2)_{t_0} \cup \bigcup_{l \in I} \mathcal{B}_{P_l}$. Moreover, for every $l \in I$, there exists $f^l$ such that

$$f^l(h(x,y,t),y,t) = x \ \ and \ \ h(f^l(z,y,t),y,t) = z$$

for every $(x,y,t) \in \mathcal{B}_{P_l}$, $(z,y,t) \in h(\mathcal{B}_{P_l})$.

In other words, each function $f^l$ is the inverse of $h$ on a small box centered on a point of the interface $\Gamma_{t_0}$. The Main Theorem 13 will be shown by using the properties of functions $f^l$ rather than those of the function $g$. For this purpose, in the next paragraph, we introduce an equivalent way to express the non-degeneracy condition I in terms of the function $f^l$ that locally is the inverse of $h$.

**Non-degeneracy condition (II)**. We say that the function $f$ defined on a domain $\mathcal{A}$ of the half space $\{\,(z,y,t) : \ z \geq 0\,\}$ such that $(0,0) \in \mathcal{A}$ satisfies the *non-degeneracy condition II* if the $p-$weighted Hessian matrix $F$ of $f$:

$$F = (F_{ij}) = \begin{pmatrix} -z^{2-p}\, f_{zz} & z^{1-p}\, f_{zy} \\ z^{1-p}\, f_{zy} & -\, f_{yy} \end{pmatrix}$$

is positive definite.

Given $g$ and $f_l$ defined above, we have:

**Proposition 14.** *The function $g$ satisfies condition I iff for every $l \in I$, the function $f_l$ satisfies condition II.*

*Proof.* For simplicity we omit the index $l$, and use $f$ instead than $f^l$. Assume that $g$ is smooth up to the interface and it satisfies condition I, then we show that I and II are equivalent. We will use the following identities:

$$f(0,y,t) = h(0,y,t),\; z^{-p}\,(f(z,y,t) - f(0,y,t)) = z^{-p}(h(z^p,y,t) - h(0,y,t)),$$

$$f_z(z,y,t) = p\, z^{p-1}\, h_z(z^p,y,t),\; f_y = h_y,\; f_{zz} = p((p-1)z^{p-2}h_z + z^{2p-2}h_{zz})$$

By the condition I we guarantee that $f$ exists. We start by observing that $f$ is continuous. The function $f^\circ = h^\circ$ is smooth by assumption. We know that $z^{-p}\,(h(z^p,y,t) - h(0,y,t)) = z^{-p}\,(f(z,y,t) - f(0,y,t))$ which implies that $\tilde{f}(z,y,t) = z^{-p}\,(f(z,y,t) - f(0,y,t))$ is smooth on $\mathcal{B}$. Similarly, $z^{1-p}\, f_z(z,y,t) = p\, h_z(z^p,y,t)$ is smooth as well. Also notice that

$$z^{2-p}\, f_{zz}(z,y,t) = p(p-1)h_z(z^p,y,t) + p\, z^p\, h_{zz}(z^p,y,t)$$

Hence $\tilde{f}_{zz}$ is smooth as well. In addition, we have the following identities:

$$h = g^q \quad h_x = q\, g^{q-1}\, g_x \quad h_y = q\, g^{q-1}\, g_y$$

$$h_{xx} = q\, g^{q-2}((q-1)g_x^2 + g\, g_{xx})$$

$$h_{yy} = q\, g^{q-2}((q-1)g_y^2 + g\, g_{yy})$$

$$h_{xy} = q\, g^{q-2}((q-1)g_y g_x + g\, g_{xy})$$

and:

$$f_z = \frac{1}{h_x} \qquad h_y = -\frac{h_y}{h_x} \qquad f_t = -\frac{h_t}{h_x} \qquad f_{zz} = -\frac{1}{h_x^3}\, h_{xx}$$

$$f_{zy} = -\frac{1}{h_x}\left(-\frac{h_y}{h_x^2}\, h_{xx} + \frac{1}{h_x} h_{xy}\right)$$

$$f_{yy} = -\frac{1}{h_x}\left(\frac{h_y^2}{h_x^2}\, h_{xx} - 2\frac{h_y}{h_x}\, h_{xy} + h_{yy}\right)$$

which yield to:

$$z^{1-p}\, f_z = \frac{h^{1-p}}{h_x} = \frac{g^{q(1-p)}}{q\, g^{q-1}\, g_x} = \frac{p}{g_x}$$

and

$$z^{2-p}\, f_{zz} = \frac{h^{2-p} h_{xx}}{h_x^3} = \frac{q^2((q-1)g_x^2 + g\, g_{xx})}{g_x^3}$$

$$z^{1-p}\, f_{zy} = -z^{1-p}\, \frac{1}{h_x}\, \left( -\frac{h_y}{h_x^2}\, h_{xx} + \frac{h_{xy}}{h_x} \right) = -p \frac{g_{xy}}{g_x^2}$$

Therefore, if $\nu$ and $\tau$ denote, respectively, the normal and tangential directions to the level sets of $g$, due to the smoothness of $g$ we can conclude that the following two matrices $F$ and $G$ are equivalent:

$$F = (F_{ij}) = \begin{pmatrix} -z^{2-p}\, f_{zz} & z^{1-p}\, f_{zy} \\ z^{1-p}\, f_{zy} & -\, f_{yy} \end{pmatrix} \simeq G = (g_{ij}) = \begin{pmatrix} (1-q)\, g_\nu^2 + g\, g_{\nu\nu} & g_{\nu\tau} \\ g_{\nu\tau} & g_{\tau\tau} \end{pmatrix}$$

□

*Remark* 10. *The Main Theorem 13 can be proven under the weaker assumption that $g \in C_s^{2+\alpha}(\Omega)$, where we say that $g \in C_s^{2+\alpha}(\Omega)$ if and only if $f_l \in C_s^{2+\alpha,p}(\mathcal{B}_l(P_l))$ for every $l \in I$.*

Next we show some key facts that we need for the proof of the Main Theorem.

Since $f$ is the inverse of $h$ and the *HMCF* is invariant under rotation, the function $f$ satisfies the fully nonlinear equation on $x > 0$:

$$f_t = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1+f_y^2) f_{xx} - 2\, f_x f_y f_{xy} - (1+f_x^2) f_{yy}} \tag{3.2}$$

By using the Inverse Function Theorem between Banach spaces one can construct a sufficiently smooth solution to this equation. If $f$ belongs to $C^{2+p}(Q_T)$ and satisfies (II) then, the equation 3.2 becomes degenerate at

$x = 0$ and, as a consequence:

$$f_t = \frac{f_{yy}}{1 + f_y^2} \quad \text{at the interface} \tag{3.3}$$

In other words, the boundary $\Gamma$ moves by the *Curve Shortening Flow.*

We introduce the set of functions $C^{k,2+\alpha,p}_{*,s}$ to be the set of all the functions in $C^{k,2+\alpha,p}_{s}$ that are strictly concave and satisfy the non-degeneracy condition I.

**Lemma 15.** *Given the operator* $\mathcal{F} : C^{k,2+\alpha,p}_{*,s} \longmapsto C^{k,\alpha,p}_{s}$*, defined as:*

$$F\,(f) = f_t - \left( \frac{\det D^2 f}{(\,1 + f_y^2\,)\, f_{xx} - 2\, f_x\, f_y\, f_{xy} + (\,1 + f_x^2\,)\, f_{yy}} \right) \tag{3.4}$$

*then, the Frechét derivative of* $\mathcal{F}$ *at the specified function* $h \in C^{k,2+\alpha,p}_{*,p}$ *is given by*

$$D\,\mathcal{F}(\,h\,)\,f = f_t - (\,x^2\, a_{11}\, f_{xx} + 2\, x\, a_{12}\, f_{xy} + a_{22}\, f_{yy} + x\, b_1\, f_x + b_2\, f_y\,) \tag{3.5}$$

*In particular, if the function* $h$ *satisfies the non-degeneracy condition II, then, the coefficients* $\{a_{ij}\}$, $b_1, b_2$ *are bounded and the matrix* $\{a_{ij}\}$ *is strictly positive. Moreover, they belong to the space* $C^{\alpha}_{s}$ *and, in particular,* $a_{22}$ *and* $b_2$ *belong to* $C^{\alpha,p}_{s}$*.*

*Proof.* It is easy to see that the coefficients are given by the following identities:

$$\begin{aligned}
a_{11} &= \frac{h_{yy}^2 + (\, h_x\, h_{yy} - h_y\, h_{xy}\,)^2 + h_{xy}^2}{x^2\, D^2} \\
a_{22} &= \frac{h_{xx}^2 + (\, h_x\, h_{xy} - h_y\, h_{xx}\,)^2 + h_{xy}^2}{D^2} \\
a_{12} &= \frac{h_{xy}(h_x h_y h_{xy} - (1+h_y^2)h_{xx}) + h_{yy}(-(1+h_x^2)h_{xy} + h_x h_y h_{xx})}{x\, D^2} \\
b_1 &= -\frac{(2h_{yy}h_x - 2h_y h_{xy})(h_{xy}^2 - h_{yy}h_{xx})}{x\, D^2} \\
b_2 &= \frac{x\,(-2h_x h_{xy} + 2h_y h_{xx})(h_{xy}^2 - h_{xx}h_{yy})}{D^2}
\end{aligned}$$

where $D := (\, 1 + h_y^2\,)\, h_{xx} - 2\, h_x\, h_y\, h_{xy} + (\, 1 + h_x^2\,)\, h_{yy}$. We observe that in the above expressions both the numerators and denominators become degenerate at $x = 0$, however, their ratios are well defined because the function $h$ is the right space. It is easy to check the boundedness of the coefficients. We need to show that the matrix $A = \{a_{ij}\}$ is positive definite, the coefficients $\{a_{ij}\}$, $b_i$ belong to $C_s^{\alpha}$ and, in particular, $a_{22}$ and $b_2$ belong to $C_s^{\alpha,p}$.

Close to $x = 0$, because of the condition II, we have:

$$a_{11} \approx \frac{x^{2-2p}\, h_{yy}^2 + (\, x^{1-p}\, h_x\, h_{yy} - h_y\, x^{1-p}\, h_{xy}\,)^2 + (x^{1-p} h_{xy})^2}{(x^{2-p}\, h_{xx})^2} \;>\; C \;>\; 0$$

$$a_{22} \approx \frac{(x^{2-p}\, h_{xx})^2 + (x^{2-p}\, h_x\, h_{xy} - h_y\, x^{2-p}\, h_{xx}\,)^2 + (x^{2-p}\, h_{xy})^2}{(x^{2-p}\, h_{xx})^2} \;>\; C \;>\; 0$$

For some constant $C > 0$. Also, it can be shown that the determinant $det(A)$ of the matrix $A$, is given by:

$$det(A) = \frac{(1 + h_x^2 + h_y^2)(h_{xy}^2 - h_{xx}\, h_{yy})^2}{x^2\, D^4}$$

and therefore, close to $x = 0$,

$$det(A) \approx \frac{x^{2-2p}(1 + h_x^2 + h_y^2)\,(x^{2-p}\,(\,h_{xy})^2 - h_{xx}\, h_{yy}))^2}{(x^{2-p}\, h_{xx})^4} \;>\; C \;>\; 0$$

The matrix $A$ is then definite positive (or equivalently the coefficients $\{a_{i,j}\}$ are uniformly elliptic).

Next we prove that $a_{11} \in C_s^{\alpha}$. By definition:

$$a_{11} \approx \frac{(x^{2-p}\, h_{yy})^2 + (x^{1-p}\, h_x\, h_{yy} - h_y\, x^{1-p}\, h_{xy}\,)^2 + x^{2-2p}\, h_{xy}^2}{(\,1 + h_y^2\,)\, x^{2-p}\, h_{xx} - 2\, x^p\, x^{1-p}\, h_x\, h_y\, x^{1-p}\, h_{xy} + (\,1 + h_x^2\,)\, x^{2-p}\, h_{yy}},$$

We have the following:

$$\{x^{1-p}\, h_x,\, x^{1-p}\, h_{xy},\, x^{2-p}\, h_{xx}\} \subseteq C_s^{\alpha} \quad \text{and} \quad \{h_y^{\circ},\, h_{yy}^{\circ}\} \subseteq C^{\alpha} \quad \Rightarrow \quad a_{11} \in C_s^{\alpha}.$$

It can be shown similarly that $a_{12}$ and $b_1 \in C_s^{\alpha}$. Next we prove that $a_{22}$ and $b_2$ belong to $C_s^{\alpha,p}$. To say that $a_{22} \in C_s^{\alpha,p}$ is equivalent to show that

$a^\circ_{22} \in C^\alpha$ and $\tilde{a}_{22} \in C^\alpha$. This is an easy task since:

$$a^\circ_{22} \;\approx\; \frac{(x^{4-2p}\,h^2_{xx})^\circ(1+h^\circ_y)}{((x^{2-p}\,h^2_{xx})^\circ(1+h^\circ_y))^2} = \frac{1}{1+(h^\circ_y)^2} \Rightarrow\; a^\circ_{22} \in C^\alpha$$

Also,

$$\tilde{a}_{22} \;=\; x^{-p}\,(a_{22} - a^\circ_{22}) \;\approx\; F\,x^{-p}\,(\,h^\circ_y - h_y\,) + G$$

with $F$ and $G$ defined by:

$$F \;:=\; \frac{h_{xx}(1+(h^\circ_y)^2)(h^2_y+(h^\circ_y)^2)}{D^2(1+(h^\circ_y)^2)}$$

$$G \;:=\; \frac{-h_{xx}\,(1+h^2_y)(-2\,h_x\,h_y\,h_{xy} + h_{yy}(1+h^2_x)) + (-2\,h_x\,h_y\,h_{xy} + h_{yy}(1+h^2_x))^2}{x^p\,D^2(1+(h^\circ_y)^2)}$$

It is easy to check that both $F$ and $G$ belong to $C^\alpha_s$, implying that $\tilde{a}_{22} \in C^\alpha$. A similar computation shows that $b_2$ belongs to $C^{\alpha,p}_s$.

□

In the next straightforward lemmas we show that the linearization of the operator $\mathcal{F}$ extended on $C^{k,2+\alpha,p}_s(Q_T)$ is invertible.

**Lemma 16.** *The operator* $L:\ C^{k,2+\alpha,p}_s(Q_T) \to C^{k,\alpha}_s(Q_T)$ *defined as:*

$$L\,(\,f\,) := f_t - (\,x^2 a_{11} f_{xx} + 2\,x\,a_{12} f_{xy} + a_{22} f_{yy} + b_1 x\,f_x + b_2 f_y + c\,f\,) \quad (3.6)$$

*such that its coefficients* $\{a_{i,j}\}$ *are uniformly elliptic,* $\{a_{i,j},\, b_i,\, c\} \subseteq C^{k,2+\alpha}_s$ *and* $\{a_{22},\, b_2,\, c\} \subseteq C^{k,2+\alpha,p}_s$ *is a continuous linear map.*

*Proof.* First all all we show that the map above is well defined. Let $f$ be in $C_s^{k,2+\alpha,p}(Q_T)$ then we show that $L\,f$ belongs to $C_s^{k,\alpha}(Q_T)$. Indeed, by definition of $C_s^{k,2+\alpha,p}(Q_T)$: $x^2\,f_{xx} \in C_s^{k,\alpha,p}$ and $x^{2-p}\,f_{xx} \in C^0$, hence $(x^2\,f_{xx})^\circ \equiv 0$. Which implies that $\widetilde{a_{22}\,x^2\,f_{xx}} = x^{-p}\,(a_{22}\,x^2\,f_{xx}) \in C^{k,\alpha}$. Analogously, $x\,a_{12}\,f_{xy}, x\,b_1\,f_x \in C_s^{k,\alpha,p}$. We use the fact that $a_{22} \in C_s^{k,\alpha,p}$ to show that $a_{22}\,f_{yy} \in C_s^{k,\alpha,p}$. Readily, all the other terms belong to $C_s^{k,\alpha,p}$.

Next, we need to prove that $L$ is a continuous linear map or analogously we show that it is bounded. But this is an easy consequence of its definition:

$$||\,L\,f\,||_{C_s^{k,\alpha,p}} \leq C\,||\,f\,||_{C_s^{k,2+\alpha,p}}$$

where the constant $C$ depends on the norm of the coefficients of the operator $L$. □

**Lemma 17.** *Let $f$ be a solution to the equation 3.6. Let $f^\circ$ and $\tilde{f}$ be defined as in Section 2.2. Then,*

$$\begin{aligned} i.\quad f^\circ_t &= a^\circ_{22} f^\circ_{yy} + b^\circ_2\, f^\circ_y + c^\circ\, f^\circ \\ ii.\quad \tilde{f}_t &= a_{11}\tilde{f}_{zz} + 2\,a_{12}\tilde{f}_{zy} + a_{22}\tilde{f}_{yy} + (a_{11}(2p-1) + b_1)\,\tilde{f}_z + (p\,a_{12} + b_2\,)\,\tilde{f}_y \\ &+ \;(-p(1-p)\,a_{11} \;+ p\,b_1 + \tilde{c}\,)\,\tilde{f} + \tilde{a}_{22}\,f^\circ_{yy} + \tilde{b}_2\,f^\circ_y + \tilde{\phi} \end{aligned}$$

*where $z = lnx$ and*

$$\begin{aligned} \hat{a}_{ij}(z,y,t) &:= a_{ij}(x,y,t) \\ \hat{b}_1(z,y,t) &:= (2p-1)\,a_{11}(z,y,t) + b_1(x,y,t) \\ \hat{b}_2(z,y,t) &:= b_2(x,y,t) \end{aligned}$$

$$\begin{aligned}\hat{c}(z,y,t) &:= e^{-p\,z}[p^2\,\hat{a}_{11}(z,y,t) - 2\,p\,\hat{a}_{12}(z,y,t) + p\,b_1(x,y,t)]\\ \hat{G}(z,y,t) &:= \tilde{b_2}(z,y,t)\,g^\circ_y(y,t) + \hat{a}_{22}(z,y,t)g^\circ_{yy}(y,t)\end{aligned}$$

*Proof.* Part i) is obvious. Part ii). Given $\tilde{f}(z,y,t) := e^{-pz}\,(f(x,y,t) - f^\circ(y,t))$, then:

$$\begin{aligned}\tilde{f}_t &= e^{-pz}\,(f_t(x,y,t) - f^\circ_t(y,t)) & \tilde{f}_z &= -p\,\tilde{f} + e^{(1-p)z}\,f_x\\ \tilde{f}_y &= e^{-pz}\,(f_y(x,y,t) - f^\circ_y(y,t)) & \tilde{f}_{yy} &= e^{-pz}\,(f_{yy}(x,y,t) - f^\circ_{yy}(y,t))\\ \tilde{f}_{zy} &= -p\,\tilde{f}_y + e^{(1-p)z}\,f_{xy}\\ \tilde{f}_{zz} &= (1-2p)\,\tilde{f}_z + p(1-p)\tilde{f} + e^{(2-p)z}\,f_{xx}\end{aligned}$$

Thus,

$$\begin{aligned}x^{-p}\,f_t &= \tilde{f}_t - \tilde{f}^\circ_t, & x^{1-p}\,f_x &= p\,\tilde{f} + \tilde{f}_z,\\ x^{-p}\,f_y &= \tilde{f}_y - \tilde{f}^\circ_y, & x^{2-p}\,f_{xx} &= \tilde{f}_{zz} + (2p-1)\,\tilde{f}_z - p(1-p)\,\tilde{f},\\ x^{-p}\,f_{xy} &= \tilde{f}_{zy} + p\,\tilde{f}_y, & x^{-p}\,f_{yy} &= \tilde{f}_{yy} - \tilde{f}^\circ_{yy}\end{aligned}$$

By substituting $f$ the last equations (3.6):

$$\tilde{f}_t(z,y,t) = x^{-p}\,(f_t(x,y,t) - f^\circ_t(y,t)) =$$

$$\begin{aligned}&a_{11}\,x^{2-p}\,f_{xx} + a_{12}\,x^{1-p}\,f_{xy} + a_{22}\,x^{-p}\,f_{yy} + b_1\,x^{1-p}\,f_x + b_2\,x^{-p}\,f_y +\\ &c\,x^{-p}\,f + x^{-p}\phi - (x^{-p}\,a^\circ_{22}\,f^\circ_{yy} + x^{-p}\,b^\circ_2\,f^\circ_y + c\,f^\circ + \phi^\circ) =\end{aligned}$$

$$a_{11}(\tilde{f}_{zz} + (2p-1)\,\tilde{f}_z - p(1-p)\tilde{f}) + 2\,a_{12}(\tilde{f}_{zy} + p\,\tilde{f}_y) + a_{22}(\tilde{f}_{yy} - x^{-p}\,f^\circ_{yy}) +$$

$$b_1(p\,\tilde f + \tilde f_z\,) + b_2(\tilde f_y - x^{-p}\,f^\circ_y) + c\,(\tilde f - x^{-p}\,f^\circ) +$$
$$+x^{-p}\phi - (x^{-p}\,a^\circ_{22}\,f^\circ_{yy} + x^{-p}\,b^\circ_2\,f^\circ_y + x^{-p}\,c\,f^\circ + x^{-p}\,\phi^\circ) =$$

$$a_{11}\tilde f_{zz} + 2\,a_{12}\tilde f_{zy} + a_{22}\tilde f_{yy} + (a_{11}(2p-1) + b_1)\,\tilde f_z + (p\,a_{12} + b_2\,)\,\tilde f_y +$$
$$(-p(1-p)\,a_{11}\, + p\,b_1 + \tilde c\,)\,\tilde f + \tilde a_{22}\,f^\circ_{yy} + \tilde b_2\,f^\circ_y + \tilde\phi$$

$\square$

In the next paragraph we denote by $\mathcal{S}_0$ the half space $x \geq 0$ in $\mathbb{R}^2$, by $\mathcal{S}$ the space $\mathcal{S} = \mathcal{S}_0 \times [0,\infty)$, and by $\mathcal{S}_T$ the space $\mathcal{S} \times [0,T]$, for $T > 0$.

Given the operator $L$ defined as above, based on the previous lemma we define the operators $L_0$ and $\tilde L$ as follows:

$$L_0\,f^\circ = f^\circ_t - (a^\circ_{22}f^\circ_{yy} + b^\circ_2\,f^\circ_y + \,c^\circ\,f^\circ\,) \tag{3.7}$$

$$\tilde L\,\tilde f = \tilde f_t - (\hat a_{11}\tilde f_{zz} + 2\,\hat a_{12}\tilde f_{zy} + \hat a_{22}\tilde f_{yy} + \hat b_1\,\tilde f_z + \hat b_2\,\tilde f_y + \hat x c\,\tilde f + \hat G) \tag{3.8}$$

where the coefficients are defined as in Lemma 17.

*Remark* 11. *It can be easily checked that* $(L\,f)^\circ = L_0\,f^\circ$ *and* $\widetilde{L\,f} = \tilde L\,\tilde f$. *We also observe that by definition the function* $g = L\,f$ *belongs to* $C^{\alpha,p}_s$. *Hence,* $g^\circ \in C^\alpha$ *and* $\tilde g \in C^\alpha$ *which implies that* $g(x,y,t) = g^\circ(y,t) + x^p\,\tilde g(ln\,x,y,t)$.

**Theorem 18.** *(Existence and Uniqueness) Let* $k$ *be a nonnegative integer and let* $\alpha$ *be a number in* $0 < \alpha < 1$. *Assume that* $\phi \in \mathcal{C}^{k,\alpha,p}_s(\mathcal{S})$ *and*

*$f^\circ \in \mathcal{C}_s^{k,2+\alpha,p}(\mathcal{S}_0)$, both $\phi$ and $f_0$ compactly supported in $\mathcal{S}$ and $\mathcal{S}_0$ respectively. Then, for any $T > 0$, the initial value problem*

$$\begin{cases} L\,f &= \phi \quad \text{in } S_T \\ f(\cdot,0) &= f_0 \quad \text{on } S_0 \end{cases} \tag{3.9}$$

*admits a unique solution $f \in C_s^{k,2+\alpha,p}(\mathcal{S}_T)$. Moreover*

$$||f||_{C_s^{k,2+\alpha,p}(\mathcal{S}_T)} \leq C(T)\left(||f_0||_{\mathcal{C}_s^{k,2+\alpha,p}(\mathcal{S}_0)} + ||\phi||_{\mathcal{C}_s^{k,\alpha}(\mathcal{S})}\right) \tag{3.10}$$

*for some constant $C(T)$, depending only $\alpha$, $k$ and $T$.*

*Proof.* It is easy to observe that the interesting case holds when the Lebesgue measure of the supports of $f_0^\circ$ and $\phi^\circ$: $|(Supp\, f_0)^\circ| \geq \eta$, $|(Supp\, \phi)^\circ| \geq \eta$, for some $\eta > 0$. Moreover, to solve the above Cauchy problem is equivalent to solve the following Cauchy problems (3.11) and (3.12).

The two problems (3.11) and (3.12) are defined as follows: Problem (3.11) is obtained by evaluating (3.9) at $x = 0$.

$$\begin{cases} L_0\, f^\circ = \phi^\circ \qquad \text{in } \mathbb{R} \times [0,T] \\ f^\circ(\cdot,0) = (f_0)^\circ \qquad \text{on } \mathbb{R} \end{cases} \tag{3.11}$$

where $L_0$ is defined as in (3.7).

The second problem is obtained by solving the corresponding problem for $\tilde{f}$.

$$\begin{cases} \tilde{L}\,\tilde{f} = \tilde{\phi} & in\ S_T \\ \tilde{f}(\cdot,0) = \tilde{f}_0 & on\ S_0 \end{cases} \tag{3.12}$$

where the operator $\tilde{L}$ is defined as in (3.8).

By the assumptions on the operator $L$ it is clear that the coefficients of the two operators $L_0$ and $\tilde{L}$ satisfy classical conditions as seen in Lemma 17. The way to proceed is the following: At first we find the solution $f^\circ$ to (3.11), then we solve (3.12). By classical theory both problems have a unique solution. Moreover, the following inequalities hold:

$$\begin{aligned} ||f^\circ||_{C^{k,2+\alpha}(\mathbb{R}^+\times[0,T])} &\leq C(T)\left(||f_0^\circ||_{C^{k,2+\alpha}(\mathbb{R}^+)} + ||g^\circ||_{\mathcal{C}^{k,\alpha}(\mathbb{R}^+)}\right) \\ ||\tilde{f}||_{C^{k,2+\alpha,(k+2)/2+\alpha/2}(\tilde{\mathcal{S}}_T)} &\leq C(T)\left(||\tilde{f}_0||_{C^{k,2+\alpha,(k+2)/2+\alpha/2}(\tilde{\mathcal{S}}_0)} + ||\tilde{\phi}||_{C^{k+\alpha,k/2+\alpha/2}(\tilde{\mathcal{S}})}\right) \end{aligned}$$

We define $f$ by $f(x,y,t) := f^\circ(y,t) + x^p\,\tilde{f}(ln\,x,y,t)$. We show that $f$ is a solution to (3.9).

Indeed, at $t = 0$, $f(\cdot,0) = f^\circ(\cdot,0) + x^p\,\tilde{f}(\cdot,0) = f_0$. Next, we show that $L\,f = \phi$. This is a simple calculation:

$$L\,f = (L\,f)^\circ + x^p\,(\widetilde{L\,f}) = L_0(\,f^\circ) + x^p\,\tilde{L}(\,\tilde{f}) = \phi^\circ + x^p\,\tilde{\phi} = \phi$$

As a consequence, the function $f$ is a solution to (3.9).

It follows as well that the solution to (3.9) is unique and it satisfies the inequality (3.10).

$\square$

Let $0 < r < 1$. We denote by $\mathcal{B}_r(P)$ the box

$$\mathcal{B}_r(P) = \left\{ \begin{pmatrix} x \\ y \\ t \end{pmatrix} : \begin{array}{c} x \geq 0, |x - x_0| \leq e^r \\ |y - y_0| \leq r \\ t_0 - r^2 \leq t \leq t_0 \end{array} \right\}$$

around the point $P = \begin{pmatrix} x_0 \\ y_0 \\ t_0 \end{pmatrix}$ and we let $\mathcal{B}_r$ be the box around the point $P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

*Remark* 12. *The choice of the box $\mathcal{B}$ is made so that it has the right rescaling. Moreover, the functionals $L_0$ and $\tilde{L}$ are well understood on the corresponding boxes $\mathcal{B}_r^\circ$ and $\tilde{\mathcal{B}}_r$.*

We have the following fundamental estimate:

**Theorem 19.** *(Schäuder Estimates) Assume that all the coefficients of the operator*

$$Lf = f_t - (x^2\, a_{11} f_{xx} + 2\, x\, a_{12} f_{xy} + a_{22} f_{yy} + x\, b_1 f_x + b_2 f_y + \, cf)$$

*belong to the space* $C_s^\alpha(\mathcal{B}_1)$ *and the coefficients* $a_{22}, b_2, c$ *belong to* $C_s^{\alpha,p}$ *for some numbers* $\alpha$, $p$ *in* $0 < p < 1$, $\alpha \leq p$ *and satisfy*

$$a_{ij}\xi^i\xi^j \geq \lambda|\xi|^2, \, \forall \xi \in \mathcal{R}^2 \setminus \{0\}$$

$$||a_{ij}||_{C_s^\alpha(Q_T)}, \, ||b_i||_{C_s^\alpha(Q_T)}, \, ||a_{22}||_{C_s^{\alpha,p}(Q_T)}, ||b_2||_{C_s^{\alpha,p}(Q_T)}, ||c||_{C_s^{\alpha,p}(Q_T)} \leq \frac{1}{\lambda}$$

*Then, there exists a constant* $C$ *depending only on* $\alpha$, $\lambda$ *and* $p$ *such that*

$$\|f\|_{C_s^{2+\alpha,p}(\mathcal{B}_{1/2})} \leq C \left( \|f\|_{C^{0,p}(\mathcal{B}_1)} + \|Lf\|_{C_s^{\alpha,p}(\mathcal{B}_1)} \right)$$

*for all functions* $f \in C_s^{2+\alpha,p}(\mathcal{B}_1)$.

*Proof.* As before, we can look at the corresponding linear operators $L_0$ and $\tilde{L}$ defined as in Lemma 17.

$$L_0 \, f^\circ = f_t^\circ - (a_{22}^\circ f_{yy}^\circ + b_2^\circ f_y^\circ + \, c^\circ g^\circ)$$

$$\tilde{L} \, \tilde{f} = \tilde{f}_t - (x^2 \, \bar{a}_{11} \tilde{f}_{xx} + 2 \, x \, \bar{a}_{12} \tilde{f}_{xy} + \bar{a}_{22} \tilde{f}_{yy} + x \, \bar{b}_1 \tilde{f}_x + \bar{b}_2 \tilde{f}_y + \, \bar{c}\tilde{F})$$

By classical theory

$$\|f^\circ\|_{C^{2+\alpha,1+\alpha/2}(\mathcal{B}_{1/2}^\circ)} \leq C \left( \|f^\circ\|_{C^0(\mathcal{B}_1^\circ)} + \|L_0 f^\circ\|_{C^{\alpha,\alpha/2}(\mathcal{B}_1^\circ)} \right) \tag{3.13}$$

for all functions $f^\circ \in C^{2+\alpha,1+\alpha/2}(\mathcal{B}_1^\circ)$.

$$\|\tilde{f}\|_{C^{2+\alpha,1+\alpha/2}(\tilde{\mathcal{B}}_{1/2})} \leq C \left( \|\tilde{f}\|_{C^0(\tilde{\mathcal{B}}_1)} + \|\tilde{L}\tilde{f}\|_{C^{\alpha,\alpha/2}(\tilde{\mathcal{B}}_1)} \right) \tag{3.14}$$

for all functions $\tilde{f} \in C^{2+\alpha,1+\alpha/2}(\tilde{\mathcal{B}}_1)$. As a result, the combination of (3.13) and (3.14) gives the desired estimate. □

*Remark* 13. *The previous theorem can be obtained for any box $\mathcal{B}_r$, $0 < r < 1$. This is because we can substitute $f(x, y, t)$ by $f(\frac{x^r}{r}, \frac{y}{r}, \frac{t}{r^2})$.*

The Schäuder estimate can be extended easily to any integer $k$ letting to the following statement:

**Theorem 20.** *(Schäuder Estimates) Assume that all the coefficients of the operator*

$$Lf = f_t - (x^2\, a_{11} f_{xx} + 2\, x\, a_{12} f_{xy} + a_{22} f_{yy} + x\, b_1 f_x + b_2 f_y + \, cf)$$

*belong to the space $C_s^{k,\alpha}(\mathcal{B}_1)$ and the coefficients $a_{22}, b_2, c$ belong to $C_s^{k,\alpha,p}$ for some numbers $\alpha$, $p$ in $0 < p < 1$, $\alpha \leq p$ and satisfy*

$$a_{ij}\xi^i\xi^j \geq \lambda|\xi|^2, \forall \xi \in \mathcal{R}^2 \setminus \{0\}$$

$$||a_{ij}||_{C_s^{k,\alpha}(Q_T)},\ ||b_i||_{C_s^{k,\alpha}(Q_T)},\ ||a_{22}||_{C_s^{k,\alpha,p}(Q_T)}, ||b_2||_{C_s^{k,\alpha,p}(Q_T)}, ||c||_{C_s^{k,\alpha,p}(Q_T)} \leq \frac{1}{\lambda}$$

*Then, there exists a constant* $C$ *depending only on* $\alpha$, $\lambda$, $k$ *and* $p$ *such that*

$$\|f\|_{C_s^{2+k\alpha,p}(\mathcal{B}_{1/2})} \leq C\left(\|f\|_{C^{k,p}(\mathcal{B}_1)} + \|Lf\|_{C_s^{k,\alpha,p}(\mathcal{B}_1)}\right)$$

*for all functions* $f \in C_s^{2+k+\alpha,p}(\mathcal{B}_1)$*.*

## 3.3 The Degenerate Equation

In this section we will extend the existence and uniqueness the Theorem 18 to a certain class of linear degenerate equations of the form

$$w_t = a^{ij} w_{ij} + b^i\, w_i + c\, w$$

on the cylinder $\mathcal{D} \times [0, T)$, $T > 0$, where $\mathcal{D}$ denotes the unit disk in $\mathbb{R}^2$. The sub-indices $i, j \in \{x, y\}$ denote differentiation with respect to the space variables $x, y$ and the summation convention is used. The matrix $\{a^{ij}\}$ is assumed to be symmetric. Certain assumptions on the coefficients will be made so that this class of equations includes, under appropriate change of coordinates, the equations (3.6).

We define the distance function $s$ in $\mathcal{D}$ as follows: In the interior of $\mathcal{D}$, $s$ it is equivalent to the standard Euclidean distance, while around any boundary point $P \in \partial\mathcal{D}$, $s$ is defined as the pull back of the distance function induced by the metric

$$ds^2 = \frac{dx^2}{x^2} + dy^2$$

on the half space $\mathcal{S}_0 = \{(x, y) : x \geq 0\}$, via a map $\varphi : \mathcal{S}_0 \cap \mathcal{D} \to \mathcal{D}$ that flattens the boundary of the disk $\mathcal{D}$ near $P$.

The parabolic distance is defined, as usual, by

$$s\left[\begin{pmatrix} P_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} P_2 \\ t_2 \end{pmatrix}\right] = s(P_1, P_2) + \sqrt{|t_1 - t_2|}.$$

We can define now the spaces $C_s^{\alpha,p}(\mathcal{D})$ and $C_s^{2+\alpha,p}(\mathcal{D})$: For a fixed small number $\delta$ in $0 < \delta < 1$, we write

$$\mathcal{D} = \mathcal{D}_{1-\delta/2} \cup (\bigcup_l (\mathcal{D}_\delta(P_l) \cap \mathcal{D}))$$

for finite many points $P_l \in \partial\mathcal{D}$, $l \in I$, with $\mathcal{D}_{1-\delta/2}$ denoting the disk centered at the origin of radius $1 - \delta/2$ and $\mathcal{D}_\delta(P_l)$ denoting the disk of radius $\delta$ centered at $P_l$.

Denote by $\mathcal{D}_+$ the half disk

$$\mathcal{D}_+ = \{\, (x, y) \in \mathcal{D} : \; x \geq 0 \,\}.$$

We can choose charts $\Upsilon_l : \mathcal{D}_+ \to \mathcal{D}_\delta(x_l) \cap \mathcal{D}$ which flatten the boundary of $\mathcal{D}$ and such that $\Upsilon_l(0) = P_l$. Lets $\psi_l$ be a partition of unity subordinated to the cover

$$\{\, \mathcal{D}_{1-\delta/2},\ (\mathcal{D}_\delta(P_l) \cap \mathcal{D})\, \}$$

of $\mathcal{D}$. We define $C_s^{\alpha,p}(\mathcal{D})$ to be the space of all functions $w$ on $\mathcal{D}$ such that $w \in C^\alpha(\mathcal{D}_{1-\delta/2})$ and $w \circ \Upsilon_l \in C_s^{\alpha,p}(\mathcal{D}_+)$ for all $l \in I$. We also define $C_s^{2+\alpha,p}(\mathcal{D})$ to be the space of all functions $w$ on $\mathcal{D}$ such that $w \in C^{2+\alpha}(\mathcal{D}_{1-\delta/2})$ and $w \circ \psi_l \in C_s^{2+\alpha,p}(\mathcal{D}_+)$ for all $l \in I$. Here $C^\alpha$ and $C^{2+\alpha}$ denote the regular Hölder Spaces, while $C_s^{\alpha,p}(\mathcal{D}_+)$ and $C_s^{2+\alpha,p}(\mathcal{D}_+)$ denote the Hölder Spaces defined in section 2.2. Both $C_s^{\alpha,p}(\mathcal{D})$ and $C_s^{2+\alpha,p}(\mathcal{D})$ are Banach Spaces under the norms

$$||w||_{C_s^{\alpha,p}(\mathcal{D})} = ||\psi\, w||_{C^\alpha(\mathcal{D}_{1-\delta/2})} + \sum_l \, ||\psi_l\, (w \circ \Upsilon_l)||_{C_s^{\alpha,p}(\mathcal{D}_+)}$$

and

$$||w||_{C_s^{2+\alpha,p}(\mathcal{D})} = ||\psi\, w||_{C^{2+\alpha,1+\alpha/2}(\mathcal{D}_{1-\delta/2})} + \sum_l \, ||\psi_l\, (w \circ \Upsilon_l)||_{C_s^{2+\alpha,p}(\mathcal{D})}.$$

In the next lemma we show that this definition is well-posed.

**Lemma 21.** *The spaces $C_s^{\alpha,p}$ and $C_s^{2+\alpha,p}$ defined above are independent of the particular choice of the number $\delta > 0$, the points $P_l \in \partial\mathcal{D}$ and charts $\Upsilon_l$. Moreover, for any two different choices of $\delta > 0$, $P_l$ and $\Upsilon_l$ the corresponding norms are equivalent.*

*Proof.* It is just an easy consequence of the definitions. □

The above definitions can be extended in a straight forward manner to the parabolic spaces $C_s^{\alpha,p}(Q)$, $C_s^{2+\alpha,p}(Q)$, where $Q$ is the cylinder $Q = \mathcal{D} \times [0,T]$, for some $T > 0$. Before we state the main result in this section, we will give the assumptions on the coefficients of the equation

$$w_t = \, a^{ij} w_{ij} + b^i \, w_i + c \, w$$

on the cylinder $Q = \mathcal{D} \times [0,T)$.

We will first assume that for any $\delta$ in $0 < \delta < 1$, the coefficients $\{a^{ij}\}$, $b^i$ and $c$ belong to the Hölder class $C^\alpha(\mathcal{D}_{1-\delta/2} \times [0,T])$, which means that the coefficients are of the class $C^\alpha$ in the interior of $\mathcal{D}$. For a number $\delta$ in $0 < \delta < 1$, let $\Upsilon_l : \mathcal{D}_+ \to \mathcal{D}_\delta(P_l) \cap \mathcal{D}$ be the collection of charts which flatten the boundary of $\mathcal{D}$, considered above. We will assume that there exists a number $\delta$ so that for every $l \in I$, the coordinate change introduced by each of the $\Upsilon_l$ transforms the operator

$$Lw = w_t - ( \, a^{ij} \, w_{ij} + b^i \, w_i + c \, w \, ) \tag{3.15}$$

on $\mathcal{D}_\delta(P_l) \cap \mathcal{D}$, into an operator $\widetilde{L}_l$ on $\mathcal{D}_+$ of the form

$$\widetilde{L}_l \, \tilde{w} = \tilde{w}_t - ( \, x^2 \, \tilde{a}_{11} \, \tilde{w}_{xx} + 2 \, x \, \tilde{a}_{12} \, \tilde{w}_{xy} + \tilde{a}_{22} \, \tilde{w}_{yy} + x \, \tilde{b}_1 \, \tilde{w}_x + \tilde{b}_2 \, \tilde{w}_y \, + \tilde{c} \, \tilde{w})$$

with the coefficients $\tilde{a}_{ij}$, $\tilde{b}_i$ and $\tilde{c}$ belonging to the class $C_s^{k,\alpha}(\mathcal{D}_+)$, with

$a_{22}$, $b_2\, c \,\in\, C_s^{k,\alpha,p}$ and

$$\tilde{a}_{ij}\xi^i\xi^j \geq \lambda\,|\xi|^2, \qquad \forall\xi \in \mathcal{R}^2 \setminus \{0\}$$

for some number $\lambda > 0$.

We next show the Hölder interpolation inequalities for these weighted spaces. In order of doing so, we define the weighted $L^\infty$ -norm:

$$||\, g \,||_{C^{k,p}(Q_\delta)} := ||\, g^\circ \,||_{C^{k,0}(Q_\delta^\circ)} + ||\, \tilde{g} \,||_{C^{k,0}(\tilde{Q}_\delta)}.$$

Also let $\vartheta(P)$ be a function which is smooth on $\mathcal{D}$, strictly positive in its interior, with

$$\vartheta(P) = dist(P, \partial\mathcal{D})$$

**Lemma 22.** *(Hölder Interpolation). For every $\epsilon > 0$ there exists a constant $C(\epsilon)$ depending on $\epsilon$, $p$, $k$ and $\alpha$ such that for any $g \in C_s^{k,2+\alpha,p}(Q_\delta)$, the following inequality holds:*

$$||\vartheta\, D\, g\,||_{C_s^{k,\alpha,p}(Q_\delta)} \leq \epsilon\, ||\, g\,||_{C_s^{k,2+\alpha,p}(Q_\delta)} + C(\epsilon)\, ||\, g\,||_{C^{k,p}(Q_\delta)}. \tag{3.16}$$

*Proof.* We prove that lemma at first for $k = 0$. By definition,

$$||\, \vartheta\, D\, g||_{C_s^{\alpha,p}(Q_\delta)} = ||\, \vartheta\, x^{-p}\, D\, \tilde{g}||_{C_s^{\alpha}(\tilde{Q}_\delta)} \leq ||\, D\, \tilde{g}_s||_{C^{\alpha}(\tilde{Q}_\delta)} + ||\, x^{1-p}\, D\, \tilde{g}_y||_{C^{\alpha}(\tilde{Q}_\delta)}$$

By standard interpolation, for any $n$ positive integer, given:

$$\bar{Q}_{n,\delta} = ([-n-1,-n] \times \mathbb{R} \times [0,T]) \cap \bar{Q}_\delta$$

by the standard Hölder inequality we have that:

$$\|\, \tilde{g} \,\|_{C^{1+\alpha,\alpha/2}(\bar{Q}_{n,\delta})} \leq \epsilon \,\|\, \tilde{g} \,\|_{C^{2+\alpha,1+\alpha/2}(\bar{Q}_{n,\delta})} + C_n(\epsilon) \,\|\, \tilde{g} \,\|_{C^0(\bar{Q}_{n,\delta})}$$

where the constants $C_n(\epsilon)$ depend only on the size of the cylinder $\bar{Q}_{n,\delta}$, thus, there exists $C$ depending on the size of $Q_{1,\delta}$ such that

$$\|\, \tilde{g} \,\|_{C^{1+\alpha,\alpha/2}(\tilde{Q}_\delta)} \leq \epsilon \,\|\, \tilde{g} \,\|_{C^{2+\alpha,1+\alpha/2}(\tilde{Q}_\delta)} + C(\epsilon) \,\|\, \tilde{g} \,\|_{C^0(\tilde{Q}_\delta)}. \tag{3.17}$$

and

$$\|\, x^{1-p}\, g^\circ \,\|_{C^{1+\alpha,\alpha/2}(Q^\circ_\delta)} \leq \epsilon \,\|\, g^\circ \,\|_{C^{2+\alpha,1+\alpha/2}(Q^\circ_\delta)} + C(\epsilon) \,\|\, g^\circ \,\|_{C^0(Q^\circ_\delta)}. \tag{3.18}$$

By taking the sum of (3.17) and (3.18), we obtain the equation (3.16) with $k = 0$. We can easily generalize the previous proof to any positive integer $k > 0$. $\square$

**Lemma 23.** *Assume that* $g \in C_s^{\alpha,p}(S_T)$ *and* $f \in C_s^{2+\alpha,p}(S)$, *for some number* $\alpha$ *in* $0 < \alpha < 1$, $p \geq \alpha$, $T > 0$. *Then, there exists a function* $h \in C_s^{2+\alpha,p}(S_T)$

*such that*

$$h(x,y,0) = f(x,y) \qquad \textit{and} \qquad \frac{\partial h}{\partial t}(x,y,0) = g(x,y,0)$$

*and*

$$||h||_{C_s^{2+\alpha,p}(S_T)} \leq C \, \Big( ||f||_{C_s^{2+\alpha,p}(S)} + ||g||_{C_s^{\alpha,p}(S_T)} \Big)$$

*for some constant $C$ depending only on $\alpha$.*

*Proof.* Given $g \in C_s^{\alpha,p}(S_T)$ and $f \in \mathcal{C}_s^{2+\alpha,p}(S)$, there exist $g^\circ \in \mathcal{C}^{\alpha,\alpha frm-e}(S^\circ_T)$, $f^\circ \in \mathcal{C}^{2+\alpha}(S^\circ)$ and $\tilde{g} \in C^{\alpha,\alpha/2}(\tilde{S}_T)$, $\tilde{f} \in C^{2+\alpha}(\tilde{S})$. Moreover, there exists $h^\circ \in C^{2+\alpha,1+\alpha/2}(S^\circ)$ such that:

$$h^\circ(y,0) = f^\circ(y) \qquad \text{and} \qquad \frac{\partial h^\circ}{\partial t}(y,0) = g^\circ(y,0)$$

and

$$||h^\circ||_{C^{2+\alpha,1+\alpha/2}(S^\circ_T)} \leq C \, \big( ||f^\circ||_{C^{2+\alpha,1+\alpha/2}(S^\circ)} + ||g^\circ||_{C^{\alpha}(S^\circ_T)} \big)$$

for some constant $C$ depending only on $\alpha$.

Moreover, there exists $\tilde{h} \in C^{2+\alpha,1+\alpha/2}(\tilde{S}_T)$ such that

$$\tilde{h}(x,y,0) = \tilde{f}(x,y) \qquad \text{and} \qquad \frac{\partial \tilde{h}}{\partial t}(x,y,0) = \tilde{g}(x,y,0)$$

and

$$||\tilde{h}||_{C_s^{2+\alpha,1+\alpha/2}(\tilde{S}_T)} \leq C \, \Big( ||\tilde{f}||_{C_s^{2+\alpha,p}(\tilde{S})} + ||\tilde{g}||_{C_s^{\alpha,p}(\tilde{S}_T)} \Big)$$

for some constant $C$ depending only on $\alpha$.

We use $h^\circ$ and $\tilde{h}$ to define:

$$h(x,y,t) = h^\circ + x^p \tilde{h}(z,y,t) \qquad\qquad z = ln(x).$$

The function $h$, hence, belongs to $C_s^{2+\alpha,p}(S_T)$ and satisfies the properties of Lemma 23.

□

**Theorem 24.** *Assume that the operator $L$ satisfies all the above conditions on the cylinder $Q = \mathcal{D} \times [0,T]$. Then, given any function $w^0 \in C_s^{k,2+\alpha,p}(\mathcal{D})$ and any function $g \in C_s^{k,\alpha,p}(Q)$ there exists a unique solution $w \in C_s^{k,2+\alpha,px}(Q_T)$ of the initial value problem*

$$\begin{cases} Lw = g & in\ Q \\ w(\cdot,0) = w^0 & on\ \mathcal{D} \end{cases}$$

*satisfying*

$$||w||_{C_s^{k,2+\alpha,p}(Q)} \leq C(T) \left( ||w^0||_{C_s^{k,2+\alpha,p}(\mathcal{D})} + ||g||_{C_s^{k,\alpha,p}(Q)} \right) \tag{3.19}$$

*The constant $C(T)$ depends only on the numbers $\alpha$, $k$, $\lambda$ and $T$.*

*Proof.* We can assume, without loss of generality, that $w^0 \equiv 0$ and that $g$ is a function in $C_s^{k,\alpha,p}(Q_T)$, which vanishes at $t = 0$ due to Lemma 23.

For $\delta > 0$, set $Q_\delta = \mathcal{D} \times [0,\delta]$ and denote by $C_{s,0}^{k,2+\alpha,p}(Q_\delta)$ and $C_{s,0}^{k,\alpha,p}(Q_\delta)$

the subspaces of $C_s^{k,2+\alpha}(Q_\delta)$ and $C_s^{k,\alpha}(Q_\delta)$ respectively, consisting out of all functions which vanish identically at $t = 0$. Also, denote by $I$ the identity operator on $C_{s,0}^{k,\alpha,p}(Q_\delta)$. We will show that, if $\delta$ is sufficiently small, there exists an operator $M : C_{s,0}^{k,\alpha,p}(Q_\delta) \to C_{s,0}^{k,2+\alpha,p}(Q_\delta)$ such that

$$|| \, L \, M - I \, || \leq \frac{1}{2}.$$

This will immediately imply that the operator $L\,M : C_{s,0}^{k,\alpha,p}(Q_\delta) \to C_{s,0}^{k,\alpha,p}(Q_\delta)$ is invertible and therefore $L : C_{s,0}^{k,2+\alpha,p}(Q_\delta) \to C_{s,0}^{k,\alpha,p}(Q_\delta)$ will be onto, as desired.

We begin by expressing the compact domain $\mathcal{D}$ as the finite union

$$\mathcal{D} = \mathcal{D}_0 \cup \bigcup_{l \geq 1} \mathcal{D}_l$$

of compact domains in such a way that

$$\text{dist}\,(\mathcal{D}_0, \partial\mathcal{D}) \geq \frac{\rho}{2} > 0$$

and for all $l \geq 1$

$$\mathcal{D}_l = B_\rho(x_l) \cap \mathcal{D}$$

with $B_\rho(x_l)$ denoting the ball centered at $x_l \in \partial\mathcal{D}$ of radius $\rho > 0$. The number $\rho > 0$ will be determined later.

The operator $L$ is non-degenerate when restricted on the interior domain $\mathcal{D}_0$. Therefore, the classical Schäuder theory for linear parabolic equations implies that $L$ is invertible when restricted on functions which vanish outside $\mathcal{D}_0$. Notice that our Hölder spaces with respect to the cycloidal metric $s$ on the interior domain $\mathcal{D}_0$ coincide with the standard Hölder spaces, where the classical Schäuder theory holds true.

We denote by $M_0 : C^{k,\alpha,p}_{s,0}(\mathcal{D}_0 \times [0,\delta]) \to C^{k,2+\alpha,p}_{s,0}(\mathcal{D}_0 \times [0,\delta])$ the inverse of the operator $L$ restricted on $\mathcal{D}_0$. Next, we consider the domains $\mathcal{D}_l$, $l \geq 1$, close to the boundary of $\mathcal{D}$, which can be chosen in such a way that the sets $B_{\rho/4}(x_l) \cap \mathcal{D}$ are disjoint. Denoting by $\overline{B}$ the half unit ball

$$\overline{B} = \{(x,y) \in B_1(0)\,;\;\; x \geq 0\,\}$$

and by $\overline{Q}_\delta$ the cylinder

$$\overline{Q}_\delta = \overline{B} \times [0,\delta]$$

we select smooth charts $\Upsilon_l : \overline{B} \to \mathcal{D}_l$, which flatten the boundary of $\mathcal{D}$, i.e., they map $\overline{B} \cap \{x = 0\}$ onto $\mathcal{D}_l \cap \partial\mathcal{D}$ and have $\Upsilon_l(0) = x_l$. This is possible if the number $\rho$ is chosen sufficiently small. Under the change of coordinates induced by the charts $\Upsilon_l$, the operator $L$, restricted on each $\mathcal{D}_l \times [0,\delta]$, is transformed to an operator $\bar{L}_l$ of the form

$$\bar{L}_l \bar{w} = \bar{w}_t - (\, x^2\, \bar{a}_l^{11} \bar{w}_{11} + 2\, x\, \bar{a}_l^{12} \bar{w}_{12} + \bar{a}_l^{22} \bar{w}_{22} + \; x \bar{b}_l^1\, \bar{w}_1 + \bar{b}_l^2\, \bar{w}_2 + \bar{c}_l\, \bar{w})$$

defined on $\overline{B} \times [0,\delta]$. Moreover, the charts $\Upsilon_l$ can be chosen appropriately so that the coefficients of $\bar{L}_l$ satisfy

$$\bar{a}_l^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 > 0 \qquad \forall \xi \in \mathbb{R}^2 \setminus \{0\}$$

and

$$||\bar{a}_l^{ij}||_{C_s^{k,\alpha,p}(\bar{Q}_\delta)} \quad ||\bar{b}_l^i||_{C_s^{k,\alpha,p}(\bar{Q}_\delta)} \quad ||\bar{c}_l||_{C_s^{k,\alpha,p}(\bar{Q}_\delta)} \leq 1/\,\bar{\lambda}$$

for some positive constant $\bar{\lambda}$.

Each of the operators $L_l$ has the form of the model operators studied previously. Denote by $S_0$ the half space $x \geq 0$ in $\mathbb{R}^2$ and by $S_\delta$ the space $\mathcal{S}_0 \times [0, \delta]$. Also, consider the subspace $\overline{C}_{s,0}^{k,\alpha,p}(\mathcal{S}_\delta)$ of $C_{s,0}^{k,\alpha,p}(\mathcal{S}_\delta)$, consisting out of functions which are compactly supported on $\mathcal{S}_\delta$. Then, Theorem 18 implies that for every $l = 1, 2, ...$ there is an operator $\bar{M}_l : \overline{C}_{s,0}^{k,\alpha,p}(S_\delta) \to C_{s,0}^{k,2+\alpha,p}(S_\delta)$ such that

$$L_l\,\bar{M}_l = I$$

with $I$ denoting the identity operator on $\overline{C}_{s,0}^{k,\alpha,p}(\mathcal{S}_\delta)$. Denote by $M_l$ the pull back of the operator $\bar{M}_l$ via the chart $\Upsilon_l$. Next, choose a nonnegative partition of unity $\phi_l$; $l = 0, 1, ...$ subordinated to the cover $\mathcal{D}_l$; $l = 0, 1, ...$ of $\mathcal{D}$ and also choose, for each $l \geq 0$, nonnegative, smooth bump functions $\psi_l$, $0 \leq \psi_l \leq 1$, supported in $\mathcal{D}_l$ with $\psi_l \equiv 1$ on the support of $\phi_l$. Then $\sum_{l \geq 0} \phi_l\ = 1$ and $\psi_l\,\phi_l = \phi_l$ for all $l$.

Our goal is to show that the operator $M : C_{s,0}^{k,\alpha,p}(Q_\delta) \to C_{s,0}^{k,2+\alpha,p}(Q_\delta)$

defined as

$$Mg \,=\, \sum \psi_l M_l \, \phi_l g$$

satisfies

$$||\, LM\, g \,-\, g\,||_{C^\alpha_s(Q_\delta)} < \frac{1}{2}\, ||g||_{C^\alpha_s(Q_\delta)} \qquad \forall g \in C^{k,\alpha,p}_{s,0}(Q_\delta)$$

if the cover $\{\mathcal{D}_l\}$ and $\delta$ are chosen appropriately. Indeed, we can write

$$L\, Mg - g = \sum_l L\, \psi_l M_l \, \phi_l g - \sum_l \phi_l g = \sum_l \, \psi_l \,(LM_l - I)\, \phi_l g + \sum_l \,[\, L, \psi_l \,]\, M_l \, \phi_l g$$

with $[\, L, \psi_l \,]$ denoting the commutator of $L$ and $\psi_l$. The commutator $[\, L, \psi_l \,]$ is only of first order and it can be estimated as

$$||[\, L, \psi_l \,]\, M_l \phi_l g||_{C^{k,\alpha,p}_s(Q_\delta)} \leq C \, \Big( ||\vartheta\, D(M_l \phi_l g)||_{C^{k,\alpha,p}_s(Q_\delta)} + ||M_l \phi_l g||_{C^{k,\alpha,p}_s(Q_\delta)} \Big) \,.$$

Let $\epsilon > 0$. It follows via the interpolation between these Hölder spaces that

$$||\vartheta\, D(M_l \phi_l g)||_{C^{k,\alpha,p}_s(Q_\delta)} \leq \epsilon \, ||M_l \phi_l g||_{C^{k,2+\alpha,p}_s(Q_\delta)} + C(\epsilon)\, ||M_l \phi_l g||_{C^{k,p}(Q_\delta)}.$$

However, for each $k$ we have

$$||M_l \phi_l \; g||_{C^{k,2+\alpha,p}_s(Q_\delta)} \leq C \, ||g||_{C^{k,\alpha,p}_s(Q_\delta)}$$

and therefore, since $M_l \phi_l g \equiv 0$ at $t = 0$,

$$||M_l \phi_l g||_{C^{k,p}(Q_\delta)} \leq C\,\delta\,||g||_{C_s^{k,\alpha,p}(Q_\delta)}.$$

Therefore if we choose $\delta$ sufficiently small we can make

$$\sum_l ||\,[\,L, \psi_l\,]\,M_l \phi_l g\,||_{C_s^{k,\alpha,p}(Q_\delta)} \leq \frac{1}{4}\,||g||_{C_s^{k,\alpha,p}(Q_\delta)}.$$

On the other hand we have $(\,LM_0 - I\,)\varphi_0 g = 0$, while for $l \geq 1$, we can make the norm of each of the operators $LM_l - I$ arbitrarily close to zero by choosing the diameters of the domains $\mathcal{D}_l$ sufficiently small. Wherein

$$||\,\sum_l \psi_l\,(L\,M_l - I)\,\phi_l g\,||_{C_s^{k,\alpha,p}(Q_\delta)} < \frac{1}{4}\,||g||_{C_s^{k,\alpha,p}(Q_\delta)}$$

for all $g \in C_s^{k,\alpha,p}(Q_\delta)$, if $\rho$ and $\delta$ are both sufficiently small. Combining the above estimates we obtain that

$$||\,L\,Mg - g\,||_{C_s^{k,\alpha,p}(Q_\delta)} \leq \frac{1}{2}\,||g||_{C_s^{k,\alpha,p}(Q_\delta)}$$

for all $g \in C_{s,0}^{k,\alpha,p}(Q_\delta)$, as desired. We conclude that for every $g \in C_{s,0}^{k,\alpha,p}(Q_\delta)$ there exists a function $w \in C_{s,0}^{k,2+\alpha}(Q_\delta)$ such that $Lw = g$. In addition

$$||w||_{C_s^{k,2+\alpha,p}(Q_\delta)} \;\leq\; C\,||g||_{C_s^{k,\alpha,p}(Q_\delta)} \tag{3.20}$$

with $C$ depending only on $\mathcal{D}$ and the constants $\alpha$, $k$, $\lambda$ and $T$. This shows the short time existence. The last inequality implies the long time existence since we can always extend the solution on a bigger interval. Hence, one can show that

$$||w||_{C_s^{k,2+\alpha,p}(Q)} \leq C(T)\left(||w^0||_{C_s^{k,2+\alpha,p}(\mathcal{D})} + ||g||_{C_s^{k,\alpha,p}(Q)}\right)$$

where the constant $C(T)$ depends only on the numbers $\alpha$, $k$, $\lambda$ and $T$.
The uniqueness follows readily from this last inequality. □

*Remark* 14. *The solution constructed is defined only up to the interface. In Chapter V, we show that this is a viscosity solution of the HMCF on the whole space.*

## 3.4 Global Change of Coordinates

In this section we introduce an appropriate global change of variables so that we transform the free boundary problem in an *initial value problem* of the form

$$\begin{cases} Mw = 0 & on\ \mathcal{D} \times [0,T] \\ w = w_0 & at \quad t = 0 \end{cases}$$

on the cylinder $\mathcal{D} \times [0,T]$, where $\mathcal{D} = \{(x,y); x^2 + y^2 \leq 1\}$ and

$$Mw = w_t - F(t, u, v, w, Dw, D^2 w)$$

is a fully non-linear operator which becomes degenerate at $\partial\mathcal{D}$.

We choose a surface $S$ close to $\Sigma$ such that the following properties hold: There exists $S : \mathcal{D} \to \mathbb{R}^3$ a parameterization of $S$ on the unit disk $\mathcal{D}$, $S(u, v) = (x, y, z) \in \mathbb{R}^3$ such that $y$ and $z$ are smooth on $\mathcal{D}$, and

$$x_u \sim \vartheta^{1-p},\ |x_{uv}| \leq C\,\vartheta^{1-p},\ x_{uu} \sim \vartheta^{2-p}$$

where $\vartheta$ behaves like distance to the boundary as in Section 3.3.
Let $T = (T_1, T_2, T_3)$ be smooth vector field transverse to $S$. Define the change of coordinates $\Phi : \mathcal{D} \times [-\eta, \eta] \to \mathbb{R}^3$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi \begin{pmatrix} u \\ v \\ w \end{pmatrix} = S(u, v) + wT(u, v) \tag{3.21}$$

or more explicitly

$$\begin{cases} x &= S_1(u, v) + wT_1(u, v) \\ y &= S_2(u, v) + wT_2(u, v) \\ z &= S_3(u, v) + wT_3(u, v) \end{cases}$$

Denote $x_u, x_v, x_w, y_u, y_v, y_w, z_u, z_v, z_w$ as partial derivatives of functions $x = x(u, v, w), y = y(u, v, w)$ with respect to $u, v, w$. Use similar notation for second derivatives of these functions. We begin by selecting a sufficiently small number $\delta > 0$, such that

$$T_1(u,v) = 0 \qquad \text{on} \;\; \mathcal{D} \setminus \mathcal{D}_{1-\delta}$$

denoting, as above, the transverse vector field to the surface $\mathcal{S}$. Notice that by choosing the surface $\mathcal{S}$ sufficiently close to the surface $z = f(x, y)$, we can make $\delta$ to depend only on the constant $\lambda$ where $\lambda$ defines the non-degeneracy condition I.

**Proposition 25.** *The function* $w$ *evolves by*

$$\begin{cases} Mw = 0 & \text{on } \mathcal{D} \times [0,T] \\ w = w_0 & \text{at} \quad t = 0 \end{cases}$$

*on the cylinder* $\mathcal{D} \times [0,T]$, *where* $\mathcal{D} = \{(x,y); x^2 + y^2 \le 1\}$ *and*

$$Mw = w_t - F(t, u, v, w, Dw, D^2 w)$$

*where* $F = F(t, u, v, w, Dw, D^2 w)$ *is a fully nonlinear operator. Moreover, its linearization* $\mathcal{D}\, M(\bar{w})\, w = w_t - \mathcal{D}\mathcal{F}(\bar{w})\, w$ *is of the form of the operator* $L$

*studied in Section 3.3:*

$$Lw = w_t - (\, a^{ij}\, w_{ij} + b^i\, w_i + c\, w\,) \tag{3.22}$$

*Proof.* To show that $w$ satisfies an equation of the desired form in the interior cylinder $\mathcal{D}_{1-\delta} \times [0, T]$ is straight forward. Hence, we will restrict our attention to $\mathcal{D} \setminus \mathcal{D}_{1-\delta}$. We start by expressing the first and second derivatives of $z$ with respect to $x, y, t$ in terms of the first and second derivatives of $w$ with respect to $u, v$ and $t$.

We remark that this computation holds in the interior of the unit disk $\mathcal{D}$. Since $z = f(x, y)$ is a function of $x, y$ we compute the first and second partial derivatives of $z$ with respect to $x$ and $y$ in terms of $w = g(u, v)$ seen as a function of $u, v$.

Let $A$ be the Jacobian matrix relative to the transformation of coordinates:

$$A = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let $\nabla z$ be the gradient of $z$:

$$\nabla z = \begin{pmatrix} \dfrac{\partial z}{\partial x} \\ \dfrac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

We denote by $A_1$ and $A_1$ the derivatives of the matrix $A$ with respect to $x$ and $y$:

$$A_1 = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 v}{\partial^2 x} \\ \dfrac{\partial^2 u}{\partial x \partial y} & \dfrac{\partial^2 v}{\partial x \partial y} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x \partial y} & \dfrac{\partial^2 v}{\partial x \partial y} \\ \dfrac{\partial^2 u}{\partial^2 y} & \dfrac{\partial^2 v}{\partial^2 y} \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Notice $c_1 = a_2$ , $d_1 = b_2$. Let $\nabla u$, $\nabla v$ be, respectively, the gradient of $u$ and $v$:

$$\nabla u = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\nabla v = \begin{pmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We denote as usual the basis vectors $e_1 = (1, 0)$ $e_2 = (0, 1)$ which will help us in performing the computation.

Finally we introduce the matrices $B_1$ and $B_2$ which denote, respectively, the derivative of the matrix $A^{-1}$:

$$A^{-1} = \begin{pmatrix} x_u, & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

with respect to $x$ and $y$: $B_1 = \dfrac{\partial A^{-1}}{\partial x}$, $B_2 = \dfrac{\partial A^{-1}}{\partial y}$ which can be computed as:

$$B_1 = \begin{pmatrix} a\,x_{11} + b\,x_{12} & a\,y_{11} + b\,y_{12} \\ a\,x_{12} + b\,x_{22} & a\,y_{12} + b\,y_{22} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} c\,x_{11} + d\,x_{12} & c\,y_{11} + d\,y_{12} \\ c\,x_{12} + d\,x_{22} & c\,y_{12} + d\,y_{22} \end{pmatrix}$$

We use the following convention to make the reading lighter:

$$x_{11} = x_{uu},\ x_{12} = x_{uv},\ x_{22} = x_{vv},\ y_{11} = y_{uu},\ y_{12} = y_{uv},\ y_{22} = y_{vv}$$

Hence, we can evaluate the coefficients of the matrices $A_1$ and $A_2$ as follows:

$$\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = \frac{\partial}{\partial x}(\nabla u) = -A \cdot B_1 \cdot \nabla u \quad \begin{pmatrix} b_1 \\ d_1 \end{pmatrix} = \frac{\partial}{\partial x}(\nabla v) = -A \cdot B_1 \cdot \nabla v$$

$$\begin{pmatrix} a_2 \\ c_2 \end{pmatrix} = \frac{\partial}{\partial y}(\nabla u) = -A \cdot B_2 \cdot \nabla u \quad \begin{pmatrix} b_2 \\ c_2 \end{pmatrix} = \frac{\partial}{\partial y}(\nabla v) = -A \cdot B_2 \cdot \nabla v$$

Moreover, since $\nabla z = A \cdot \begin{pmatrix} z_u \\ z_v \end{pmatrix} + z_w \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix}$, we obtain that:

$$\frac{\partial}{\partial x}\nabla z = A_x \cdot \begin{pmatrix} z_u \\ z_v \end{pmatrix} + A \cdot \begin{pmatrix} a\, z_{uu} + b\, z_{uv} + z_{uw}\dfrac{\partial w}{\partial x} \\ a\, z_{uv} + b\, z_{vv} + z_{vw}\dfrac{\partial w}{\partial x} \end{pmatrix}$$

$$+ \; z_w \begin{pmatrix} \dfrac{\partial^2 w}{\partial^2 x} \\ \dfrac{\partial^2 w}{\partial x \partial y} \end{pmatrix} + (a\, z_{uw} + b\, z_{vw}) \begin{pmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{pmatrix}$$

$$\frac{\partial}{\partial y}\nabla z = A_y \cdot \begin{pmatrix} z_u \\ z_v \end{pmatrix} + A \cdot \begin{pmatrix} c\, z_{uu} + d\, z_{uv} + z_{uw}\dfrac{\partial w}{\partial y} \\ c\, z_{uv} + d\, z_{vv} + z_{vw}\dfrac{\partial w}{\partial y} \end{pmatrix}$$

$$+ \; z_w \begin{pmatrix} \dfrac{\partial^2 w}{\partial x \partial y} \\ \dfrac{\partial^2 w}{\partial^2 y} \end{pmatrix} + (c\, z_{uw} + d\, z_{vw}) \begin{pmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{pmatrix}$$

The gradient of the function $w$, as well as its partial derivatives, can be expressed by using the matrix $A$:

$$\begin{pmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} a\, w_u + b\, w_v \\ c\, w_u + d\, w_v \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{\partial^2 w}{\partial^2 x} \\ \dfrac{\partial^2 w}{\partial x \partial y} \end{pmatrix} = \begin{pmatrix} a_1\, w_u + a\,(a\, w_{uu} + b\, w_{uv}) + b_1\, w_v + b(a\, w_{uv} + b\, w_{vv}) \\ c_1\, w_u + c(a\, w_{uu} + b\, w_{uv}) + d_1\, w_v + d(a\, w_{uv} + b\, w_{vv}) \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{\partial^2 w}{\partial x \partial y} \\ \dfrac{\partial^2 w}{\partial^2 y} \end{pmatrix} = \begin{pmatrix} a_2\, w_u + a\,(c\, w_{uu} + d\, w_{uv}) + b_2\, w_v + b(c\, w_{uv} + d\, w_{vv}) \\ c_2\, w_u + c(c\, w_{uu} + d\, w_{uv}) + d_2\, w_v + d(c\, w_{uv} + d\, w_{vv}) \end{pmatrix}$$

Therefore, after performing all the substitutions above we get:

$$\frac{\partial^2 z}{\partial^2 x} = A^1_{11}\, w_{11} + A^1_{12}\, w_{12} + A^1_{22}\, w_{22} + B^1_1\, w_1 + B^1_2\, w_2 + B^1_{12}\, w_1\, w_2 + B^1_{11}\, w_1^2 + B^1_{22}\, w_2^2 + C_1$$

$$\frac{\partial^2 z}{\partial^2 y} = A^2_{11}\, w_{11} + A^2_{12}\, w_{12} + A^2_{22}\, w_{22} + B^2_1\, w_1 + B^2_2\, w_2 + B^2_{12}\, w_1\, w_2 + B^2_{11}\, w_1^2 + B^2_{22}\, w_2^2 + C_2$$

$$\frac{\partial^2 z}{\partial x \partial y} = A^\circ_{11}\, w_{11} + A^\circ_{12}\, w_{12} + A^\circ_{22}\, w_{22} + B^\circ_1\, w_1 + B^\circ_2\, w_2 + B^\circ_{12}\, w_1\, w_2 + B^\circ_{11}\, w_1^2 + B^\circ_{22}\, w_2^2 + C_\circ$$

where the coefficients $A^k_{i,j}$, $B^j_i$, $C^i$ are obtained as follows:

$$\begin{aligned}
A^1_{11} : &= -a^2\,(\,-z_w + b\, y_w(z_u + w_1\, z_w) + d\, y_w(z_v + w_2 z_w)) \\
A^1_{22} : &= -b^2\,(\,-z_w + b\, y_w(z_u + w_1\, z_w) + d\, y_w(z_v + w_2 z_w)) \\
A^1_{12} : &= -2\,a\,b\,(\,-z_w + b\, y_w(z_u + w_1\, z_w) + d\, y_w(z_v + w_2 z_w)) \\
B^1_{11} : &= -2ab(ay_{uw} + by_{vw}) \\
B^1_{22} : &= -2bd(ay_{uw} + by_{vw}) \\
B^1_{12} : &= -2(b^2 + ad)(ay_{uw} + by_{vw})
\end{aligned}$$

We look at the evolution of $w_t$. Suppose that $z = f(x, y, t)$, then differentiating with respect to t:

$$\begin{aligned}\frac{\partial w}{\partial t}=\frac{\dfrac{\partial z}{\partial t}}{z_y\,y_w-z_w} &= \gamma_{11}\,w_{11}^2+2\,\gamma_{12}\,w_{12}^2+\gamma_{22}\,w_{22}^2+\rho_{11}\,w_{11}+\rho_{12}\,w_{12}\\ &+ \rho_{22}\,w_{22}+\gamma_1\,w1^2+\gamma_2\,w2^2+\rho_1\,w1+\rho_2\,w_2+\Theta.\end{aligned}$$

By introducing this global change of variables one can transform the free boundary problem in an *initial value problem* of the form

$$\begin{cases} Mw=0 & on\ \mathcal{D}\times[0,T]\\ w=w_0 & at\quad t=0\end{cases}$$

on the cylinder $\mathcal{D}\times[0,T]$, where $\mathcal{D}=\{(x,y);x^2+y^2\leq 1\}$ and

$$Mw=w_t-F(t,u,v,w,Dw,D^2w)$$

where the fully nonlinear operator $F$ is defined as

$$\begin{aligned}F(t,u,v,w,Dw,D^2w) &= \gamma_{11}\,w_{11}^2+2\,\gamma_{12}\,w_{12}^2+\gamma_{22}\,w_{22}^2+\rho_{11}\,w_{11}+\rho_{12}\,w_{12}\\ &+ \rho_{22}\,w_{22}+\gamma_1\,w1^2+\gamma_2\,w2^2+\rho_1\,w1+\rho_2\,w_2+\Theta\end{aligned}$$

where the coefficients $\gamma_{i,j}$, $\rho_{i,j}$, $\gamma_i$, $\rho_i$, $\Theta$ are very complicated functions depending on $w_1$, $w_2$, $z_v$, $z_{vw}$ and $x_w$, $z_{vw}$, $x_{vv}$, $x_w$, $x_{vw}$, $x_{uw}$.
However, these coefficients can be estimated under the assumptions:

$$w_2\sim 0,\ z_v\sim 0,\ z_{vw}\sim 0\ \text{ and } x_w,\ z_{vw}\sim 0,\ x_{vv}\sim 0,\ x_w,\ x_{vw},\ x_{uw}\sim 0.$$

We omit here all the details but it can be shown that the linearization of the above equation at a point $\bar{w}$ close to $w_0$. By doing the substitutions above we can compute the linearization $\mathcal{L}\,\tilde{w} = \mathcal{DF}(\bar{w})\,\tilde{w}$ can be expressed by the equation:

$$\mathcal{L}(\tilde{w}) = a_{11}\,\tilde{w}_{11} + 2\,a_{12}\,\tilde{w}_{12} + a_{22}\,\tilde{w}_{22} + b_1\,\tilde{w}_1 + b_2\,\tilde{w}_2 + c\,\tilde{w} + d$$

with the coefficients of the operator $\mathcal{L}$ behave as described below:

$$\begin{array}{lcllcl} a_{11} & \approx & \dfrac{w_1^2}{w_{11}^2}\,g_{11} & \qquad a_{12} & \approx & \dfrac{w_{11}\,w_1}{w_{11}^2}\,g_{12} \\ a_{11} & \approx & \dfrac{w_{11}^2}{w_{11}^2}\,g_{22} & \qquad b_1 & \approx & \dfrac{w_{11}\,w_1}{w_{11}^2}\,h_1 \\ b_2 & \approx & \dfrac{w_{11}^2}{w_{11}^2}\,h_2 & \qquad c & \approx & \dfrac{w_{11}\,w_1}{w_{11}^2}\,h \end{array}$$

The functions $g_{i,j}$, $h_i$ belong to the space $C_s^{\alpha}$ and particularly $g_{22}$, $b_2$, $c$, $d$ belong to $C_s^{\alpha,p}$. Therefore, this shows that the linearization

$$\mathcal{D}\,M(\bar{w})\,w = w_t - \mathcal{DF}(\bar{w})\,w = w_t - \mathcal{L}\,w$$

is of the form of the operator $L$ defined and studied in Section 3.4. This concludes the proof. $\square$

## 3.5 Proof of Main Theorem

The local coordinate change, presented in Section 3.1 can't be used directly for the proof of the following theorem, since the result is global. Instead, we will use the previous global change of coordinates in the proof of the Main Theorem.

We assume that the surface $\Sigma$ is the union

$$\Sigma = \Sigma_1 \cup \Sigma_2$$

where $\Sigma_1$ is the flat side and $\Sigma_2$ is the strictly convex part of the surface and denote, as before, by $\Gamma$ the junction curve

$$\Gamma = \Sigma_1 \cap \Sigma_2.$$

We can also assume, by rotating the coordinates, that the flat side lies on the $x = 0$ plane and that $x > 0$ on $\Sigma_2$ and that the non-degeneracy condition I holds.

We pick a surface $\mathcal{S}$, sufficiently close to the surface $\Sigma_2$ as we did in Section 3.4. such that its boundary $\partial\mathcal{S}$ lies on the $x = 0$ plane.

**Definition** 26. *We say that the initial surface* $\Sigma$ *is of the class* $C_s^{k,2+\alpha,p}$ *if the function* $w^0$ *belongs to the class* $C_s^{k,2+\alpha,p}(\mathcal{D})$. *We say that* $\Sigma_t$, $0 \leq t \leq T$ *is of the class* $C_s^{k,2+\alpha,p}$ *if the function* $w(t)$ *belongs to the class* $C_s^{k,2+\alpha,p}(\mathcal{D} \times [0,T])$. *Finally, we say that* $\Sigma_t$ *is smooth up to the interface, for* $0 < t \leq T$, *if the* *function* $w(t)$ *is smooth on* $\mathcal{D} \times (0, T]$.

**Theorem 27.** *Assume that for some nonnegative integer* $k$ *and some number* $p$ *in* $0 < p < 1$, $\alpha \leq p$, *the strictly convex part* $\Sigma_2$ *of the initial surface* $\Sigma$ *belongs to the class* $C_s^{k,2+\alpha,p}$ *and satisfies condition I. Then, under the coordinate change studied in Section 3.4 the Harmonic Mean Curvature Flow*

$$\frac{\partial P}{\partial t} = \frac{K}{H} \vec{N} \qquad t \in [0, T]$$

*with initial data the surface* $\Sigma$ *converts into the initial value problem*

$$\begin{cases} Mw = 0 & (u, v, t) \in \mathcal{D} \times [0, T] \\ w(u, v, 0) = w^0 & (u, v) \in \mathcal{D} \end{cases}$$

*with* $w^0 \in C_s^{k,2+\alpha,p}(\mathcal{D})$ *and*

$$Mw = w_t - F(t, u, v, w, Dw, D^2 w)$$

*satisfies the hypotheses of Theorem 3.3 proved in Chapter III.*

*Proof.* Due to the computation of Section 3.4 we transform the free boundary problem for the Harmonic Mean Curvature Flow, into the initial value problem for the fully-nonlinear operator $M$. The condition I implies that if the time $T$ is sufficiently small, the linearization of the operator $M$ at a point $\bar{w} \in C_s^{k,2+\alpha,p}(\mathcal{D} \times [0, T])$ sufficiently close to $w^0$ satisfies all the hypotheses of Theorem 3.3: In the interior of $\mathcal{D}$ the operator $M$ is strictly parabolic, since the surface $\Sigma_2(t)$ is strictly convex away from the interface $x = 0$ and the coordinate change is smooth. On the other hand, we showed in Chapter III that locally at any point at the boundary of $\mathcal{D}$, the linearization of the operator $M$ can be transformed, under an appropriate change of variables, into an operator of the form

$$f_t = x^2 a_{11} f_{xx} + 2\, x\, a_{12} f_{xy} + a_{22} f_{yy} + b_1 x\, f_x + b_2 f_y \tag{3.23}$$

defined on the half disk $\mathcal{D}_+ = \mathcal{D} \cap \{x \geq 0\}$ and satisfying the hypotheses of Theorem 3.3. This has been shown in details in Chapter III, for the local change of coordinates. The substitution $g = h^p$ is included in the coordinate change 3.4, since the initial surface, close to the interface $\Gamma$ is the graph of a function which vanishes as distance to the boundary to the power $q = 1/p \geq 1$ at $z = 0$ and the vector field $T = (T_1, T_2, T_3)$ is taken to be parallel to the $z = 0$ plane when $0 \leq z \leq \delta$. Hence, the Inverse Function Theorem between Banach spaces applies here and we conclude the proof of the theorem.

□

## 3.6 Motion of the Free Boundary

**Theorem 28.** *Let $\Sigma_t$ be a family of surfaces of class $\mathfrak{S}$ which is a solution to the Harmonic Mean Curvature Flow. Let $\Gamma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$ be the interface between the flat side $(\Sigma_1)_t$ and the strictly convex side $(\Sigma_2)_t$, then $\Gamma_t$ evolves* *by the Curve Shortening Flow.*

$$\frac{\partial Q}{\partial t} = k \ \vec{n} \tag{3.24}$$

*where $Q$ is a point of the curve $\Gamma_t$, $k$ is its the curvature at the point $Q$ and $\vec{n}$ is the normal to the curve $\Gamma_t$ at the point $Q$.*

*Proof.* Let $P$ be a point on the boundary $\Gamma_t$. We can assume without loss of generality that $\Gamma_t$ lies on the $x = 0$ plane. Then, we can write the surface $\Sigma_t$ around the point $P$ as the graph a function $f$. The function $f$ belongs to the space $C_s^{2+\alpha,p}$ and it evolves by the degenerate equation 3.2, reproduced below.

$$f_t = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_y^2) f_{xx} - 2\, f_x f_y f_{xy} - (1 + f_x^2) f_{yy}} \tag{3.25}$$

Moreover it satisfies the non-degeneracy condition II. □

As $x$ approaches 0, $f_{xx}$ goes to infinity with rate $x^{p-2}$, with $0 < p < 1$, while, $f_x$, $|f_{xy}|$ are bounded by $x^{p-1}$ and $f_y$, $f_{yy}$ are uniformly bounded. Hence,

$$\lim_{x \to 0} f_t(x, y, t) = \lim_{x \to 0} \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_y^2) f_{xx} - 2\, f_x f_y f_{xy} - (1 + f_x^2) f_{yy}} =$$

$$= \lim_{x \to 0} \frac{f_{yy} - \dfrac{f_{xy}^2}{f_{xx}}}{(1 + f_y^2) - 2\, \dfrac{f_x f_y f_{xy}}{f_{xx}} - \dfrac{(1 + f_x^2)}{f_{xx}} f_{yy}}$$

with

$$\lim_{x\to 0}\frac{f_{xy}^2}{f_{xx}}=0,\ \lim_{x\to 0}\frac{f_x f_y f_{xy}}{f_{xx}}=0\ \text{ and }\ \lim_{x\to 0}\frac{(1+f_x^2)}{f_{xx}}f_{yy}=0$$

implying that

$$f_t(0,y,t)=\frac{f_{yy}(0,y,t)}{1+f_y^2(0,y,t)}.$$

In other words, the evolution of the point $P$ is governed by the Curve Shortening Flow.

**Corollary 29.** *The interface curve $\Gamma_t$ between the flat side and the strictly convex side becomes strictly convex and smooth at $t>0$.*

*Proof.* It is a consequence of the fact the initial curve $\Gamma$ evolves by Curve Shortening Flow. Gage and Hamilton proved in [14] that a convex curve in $\mathbb{R}^2$ evolving by Curve Shortening Flow becomes smooth and strictly convex at time $t>0$. □

# Chapter 4

# Regularity

## 4.1 Higher Regularity

In this chapter we study the regularity of solution of the *HMCF* and finally we prove the main result of the thesis.

**Theorem 30.** *Assume that the initial surface* $\Sigma$ *satisfies the assumptions of Theorem 13. Then, the solution* $\Sigma_t$ *of the HMCF is converted, via the coordinate change studied in Section 3.4, to a function* $w(t)$ *which belongs, for any positive integer* $k$*, to the Hölder class* $C_s^{k,2+\alpha,p}(Q)$*, on* $Q = \mathcal{D} \times [0,T]$*. Moreover, for any* $\tau$ *in* $0 < \tau < T$ *we have*

$$||w||_{C_s^{k,2+\alpha,p}(\mathcal{D}\times[\tau,T])} \leq C_k(\tau, ||w^0||_{C_s^{2+\alpha,p}(\mathcal{D})}) \tag{4.1}$$

*Proof.* It follows from the classical theory of non-degenerate parabolic equations that the strictly convex part $(\Sigma_2)_t$ of the surface $\Sigma_t$ is a smooth surface at the points where $x > 0$. Hence we will restrict our attention at points on the interface $\Gamma_t$. To obtain the desired regularity we will use the local change of coordinates introduced in Section 3.2. Let $P_0 = (z_0, x_0, t_0)$ be a point on the junction curve $\Gamma_t$ at time $t = t_0$ in $0 < t_0 < T$ and assume, by rotating the coordinates, that at the point $P_0$ the normal vector to $\Gamma_{t_0}$ pointing outwards the flat side $(\Sigma_1)_{t_0}$ is parallel to the $z$-axis, We assume therefore that the surface can be seen as the graph of the function $f = f(z, y, t)$, defined in a small box

$$\mathcal{B}_\eta = \{\, 0 \le z \le\, e^\eta,\, |y - y_0| \le \eta,\, -\eta^2 \le t - t_0 \le 0\}.$$

We have computed in Section 2.3 that $f$ satisfies the evolution equation

$$f_t = \frac{f_{xx}\, f_{yy} - f_{xy}^2}{(\,1 + f_y^2\,) f_{xx} - 2\, f_x f_y\, f_{xy} + (\,1 + f_x^2\,) f_{yy}}. \tag{4.2}$$

By definition the surface $\Sigma_t$ belongs to the class $C_s^{2+\alpha,p}$, and as a consequence the function $f$ belongs to the class $C_s^{2+\alpha,p}(\mathcal{B}_\eta)$, as defined in Section 2.3. Our goal is to show that $f \in C_s^{k,2+\alpha,p}$. We will prove first that $f \in C_s^{1,2+\alpha}(\mathcal{B}_\eta)$, which means that $f_t$, $f_y$ and $x\, f_x$ belong to the class $C_s^{2+\alpha}(\mathcal{B}_\eta)$. This will be shown by computing the equations satisfied by each of the functions $x\, f_x$, $f_y$ and $f_t$. We will first show that $f_y$ belongs to the class $C_s^{2+\alpha}(\mathcal{B}_\eta)$.

Differentiating equation (4.2) with respect to $y$ we obtain the following equation for $\tilde{f} = f_y$:

$$\begin{aligned}\tilde{f}_t \;=\; & \left(\frac{f_{xy}^2 + (f_y\, f_{xy} - f_{yy}\, f_x\,)^2}{D^2}\right)\tilde{f}_{xx} + \\ & 2\left(\frac{-\,f_{xy}\, f_{yy}\,(f_x^2+1) - f_{xy}\, f_{xx}(1+f_y^2) + f_y\, f_x\,(f_{xy}^2 + f_{yy}\, f_{xx})}{D^2}\right)\tilde{f}_{xy} + \\ & \quad \left(\frac{(f_{xy}\, f_x - f_y\, f_{xx})^2 + f_{xy}^2 + f_{xx}^2}{D^2}\right)\tilde{f}_{yy} + \\ & \quad 2\left(\frac{f_y\,(\,2\, f_x\, f_{yy}\, f_{xx} - f_{xy}^3)}{D^2}\right)\tilde{f}_x - 2\left(\frac{f_y\, f_{yy}\, f_{xx}^2}{D^2}\right)\tilde{f}_y.\end{aligned}$$

where $D = (\,1 + f_y^2\,)f_{xx} - 2\, f_x f_y\, f_{xy} + (\,1 + f_x^2\,)f_{yy}$.

Since function $f \in C_s^{2+\alpha,p}(\mathcal{B}_\eta)$, the above equation is of the form

$$\tilde{f}_t = x^2\, a_{11}\tilde{f}_{xx} + 2x\, a_{12}\tilde{f}_{xy} + a_{22}\tilde{f}_{yy} + x\, b_1\tilde{f}_x + b_2\tilde{f}_y$$

where all the coefficients belong to the Hölder class $C_s^\alpha$, the matrix $(a_{ij})$ is strictly positive and $a_{22}$ and $b_2$ belong to $C_s^{\alpha,p}$.

It then follows from theorem 19, appropriately rescaled, that $f_y = \tilde{f}$ belongs to the class $C_s^{2+\alpha}(\mathcal{B}_\eta)$. Similarly, it can be shown that $f_t \in C_s^{2+\alpha}(\mathcal{B}_\eta)$. We will finally show that $x\, f_x \in C_s^{2+\alpha}(\mathcal{B}_\eta)$. We compute the evolution of $\tilde{f} = x\, f_x$ by differentiating the equation (4.2) with respect to $x$ and multiplying it by $x$. We find that $\tilde{f}$ satisfies the equation:

$$\begin{aligned}
\tilde{f}_t = & \left( \frac{f_{yy}^2 + (\, f_{yy}\, f_x - f_y\, f_{xy}\,)^2 + f_{xy}^2}{D^2} \right) \tilde{f}_{xx} \\
& + 2 \left( \frac{f_x\, f_y\, (\, f_{xy}^2 + f_{xx}\, f_{yy}\,) - (1 + f_y^2)\, f_{xx}\, f_{xy} - (1 + f_x^2)\, f_{xy}\, f_{yy}}{D^2} \right) \tilde{f}_{xy} \\
& + \left( \frac{f_y^2 + (\, f_x\, f_{xy} - f_y\, f_{xx})^2 + f_{xx}^2}{D^2} \right) \tilde{f}_{yy} + 2 \left( \frac{f_x\, f_{yy}\, (+f_{xx}\, f_{yy} - f_{xy}^2)}{D^2} \right) \tilde{f}_x \\
& - 2 \left( \frac{f_x\, f_{xy}(\, f_{xx}\, f_{yy} - f_{xy}^2)}{D^2} \right) \tilde{f}_y + \left( \frac{(f_y + f_{yy})(\, f_{xx}\, f_{yy} - f_{xy}^2\,)}{D^2} \right)
\end{aligned}$$

Since the function $f$ belongs to the Hölder class $C_s^{2+\alpha,p}(\mathcal{B}_\eta)$ we can easily show that the above equation is of the form

$$\tilde{f}_t = x^2\, a_{11} \tilde{f}_{xx} + 2x\, a_{12} \tilde{f}_{xy} + a_{22} \tilde{f}_{yy} + x\, b_1 \tilde{f}_x + b_2 \tilde{f}_y + g$$

where all the coefficients and $g$ belong to the Hölder class $C_s^{\alpha,p}(\mathcal{B}_\eta)$, the matrix $(a_{ij})$ is strictly positive and all the necessary conditions are fulfilled. Once again Theorem 19, appropriately rescaled, implies that $x\, f_x = \tilde{f}$ belongs to the class $C_s^{2+\alpha,p}(\mathcal{B}_\eta)$, as desired.

*Generalization for any* $k > 1$. *By the same technique we obtain that* $f \in C_s^{k,2+\alpha,p}(\mathcal{B}_\eta)$, *for any positive integer* $k$. *Since this is true at any point* $P_0 \in \Gamma_t$, $0 < t < T$, *by changing coordinates, we can conclude that the function* $w$ *belongs to the class* $C_s^{k,2+\alpha,p}(Q)$, *on* $Q = \mathcal{D} \times [0, T]$. *Moreover, via a standard regularizing argument it can be proven that for any* $\tau$ *in* $0 < \tau < T$ *we have* *the estimate:*

$$||w||_{C_s^{k,2+\alpha,p}(\mathcal{D}\times[\tau,T])} \leq C_k(\tau, ||w^0||_{C_s^{2+\alpha,p}(\mathcal{D})}). \tag{4.3}$$

$\square$

We are now ready to prove the main theorem of the thesis.

**Theorem 31.** *Given a surface* $\Sigma$ *of class* $C_s^{2+\alpha,p}$ *that satisfies condition I, then, there exists a time* $T > 0$ *for which the HMCF*

$$\frac{\partial P}{\partial t} = \frac{K}{H}\ \vec{N}$$

*admits a solution* $\Sigma_t$ *on* $0 \leq t \leq T$ *which is smooth up to the interface* $z = 0$ *for all* $0 < t < T$*, satisfies condition I. Moreover, the interface* $\Gamma$ *between the flat side and the strictly convex side will be a strictly convex smooth curve evolving by the Curve Shortening Flow.*

*Proof.* The proof follows readily combining the Main Theorem 13, the Regularity Theorem 30 and the next theorem. $\square$

**Theorem 32.** *Let* $k$ *be a nonnegative integer,* $\alpha$, $p$ *numbers in* $0 < p < 1$, $\alpha \leq p$ *and* $T > 0$ *a positive number. Also, let* $w^0$ *be a function in* $C_s^{k,2+\alpha,p}(\mathcal{D})$*. Assume that the linearization* $DM(\bar{w})$ *of the fully-nonlinear operator*

$$Mw = w_t - F(t, x, y, w, Dw, D^2w)$$

*defined on the cylinder* $Q = \mathcal{D} \times [0, T]$*, satisfies the hypotheses of Theorem 3.3 at all points* $\bar{w} \in C_s^{k,2+\alpha,p}(Q)$*, with* $||\bar{w} - w^0||_{C_s^{k,2+\alpha,p}(Q)} \leq \mu$*, for some* $\mu > 0$*. Then, there exists a number* $\tau_0$ *in* $0 < \tau_0 \leq T$ *depending on the constants* $\alpha$*,* $p$*,* $k$*,* $\lambda$*, and* $\mu$*, for which the initial value problem*

$$\begin{cases} w_t = F(t, u, v, w, Dw, D^2 w) & \textit{in } \ \mathcal{D} \times [0, \tau_0] \\ w(\cdot, 0) = w^0 & \textit{on } \ \mathcal{D} \end{cases}$$

*admits a solution* $w$ *in the space* $C_s^{k,2+\alpha,p}(\Omega \times [0, \tau_0])$*. Moreover,*

$$||w||_{C_s^{k,2+\alpha,p}(\mathcal{D} \times [0,\tau_0])} \leq C \, ||w^0||_{C_s^{k,2+\alpha,p}(\mathcal{D})}$$

*for some positive constant* $C$ *which depends only on* $\alpha$*,* $p$*,* $k$*,* $\lambda$ *and* $\mu$*.*

*Proof.* The proof is a consequence of the Inverse Function Theorem between Banach Spaces. □

**Theorem 33.** *Under the same assumptions as in Theorem 27, there exists a number* $\tau_k > 0$ *for which the HMCF*

$$\frac{\partial P}{\partial t} = \frac{K}{H} \vec{N}, \qquad t \in [0, T]$$

*with initial data the surface* $\Sigma$ *admits a solution* $\Sigma_t$ *on* $0 \leq t \leq \tau_k$*. Moreover, under the coordinate change the strictly convex part* $(\Sigma_2)_t$ *of* $\Sigma_t$ *is converted to a function* $w(t)$ *which belongs to the Hölder class* $C_s^{k,2+\alpha,p}(Q_k)$*, on* $Q_k = \mathcal{D} \times [0, \tau_k]$*.*

Because of Theorem 30 we have that the function $w$ is smooth up to the boundary of $\mathcal{D}$ for all $t \in (0, T]$, for some $T > 0$.

## 4.2 Smoothness

In this section we will give the proof of the Main Theorem stated in Chapter III. We will actually prove the following stronger result, where we relax the regularity assumptions on the initial surface.

**Theorem 34.** *Assume that the strictly convex part* $\Sigma_2$ *of the initial surface* $\Sigma$ *belongs to the class* $C_s^{2+\alpha,p}$, *for some numbers* $p$ *in* $0 < p < 1$, $\alpha \leq p$ *and satisfies condition I. Then, the HMCF*

$$\frac{\partial P}{\partial t} = \frac{K}{H} \vec{N} \qquad t \in [0, T]$$

*with initial data the surface* $\Sigma$ *admits a solution* $\Sigma_t$ *which is smooth up to the interface, for* $0 < t \leq T$. *In particular the interface* $\Gamma_t$ *is a smooth curve for every* $0 < t \leq T$ *and it moves by the Curve Shortening Flow.*

*Proof.* It can be easily checked that if the initial surface satisfies the conditions of the Main Theorem, then it will satisfy the weaker conditions of Theorem 34. Assume that the strictly convex part $\Sigma_2$ of the initial surface $\Sigma$ belongs to the class $C_s^{2+\alpha,p}$, for some numbers $p$ in $0 < p < 1$, $\alpha \leq p$ and satisfies condition I, then we have proven existence for the *HMCF* in Theorem 27. From Theorem 30 we have that $w \in C_s^{k,2+\alpha,p}(\mathcal{D} \times (0, T])$, for all nonnegative integers $k$. In particular this implies that for all integers $k$ we have $w \in C^{k,p}(\mathcal{D} \times (0,T])$, for all $\tau$ in $0 < \tau < T$. It follows that $w$ is $C^{\infty,p}$ smooth up to the boundary of $\mathcal{D}$. Going back to the original coordinates, we conclude that the strictly convex part of the surface $\Sigma(t)$, $0 < t \leq T$ is smooth up to $z = 0$ and that the interface $\Gamma(t)$ is smooth.

$\square$

# Chapter 5

# Viscosity Solutions

## 5.1 Definition of Viscosity Solution

In this chapter we show that our solutions can be interpreted as viscosity solutions. We give at first the definition of a viscosity solution and then we show the comparison principle which is fundamental for the purpose of this chapter. Finally, we show a geometric property of the solutions to the *HMCF*.

**Definition** 35. *Given two surfaces* $\Sigma_1$ *and* $\Sigma_2$ *we say that* $\Sigma_1$ *is contained in* $\Sigma_2$ *if the region enclosed by* $\Sigma_1$ *is contained in the region enclosed by* $\Sigma_2$*.*

**Definition** 36. *A family of convex regions* $\{\Sigma_t\}_{0<t<T}$ *is called a viscosity solution of Equation 1.1 if the following conditions hold: For any smooth, strictly convex surface* $\Sigma_0^-$ *contained in* $\Sigma_{t_0}$ *for some time* $t \in (0,T)$*, the*

*surfaces* $\Sigma_t^-$ *given by solving Equation 1.1 are contained in* $\Sigma_{t_0+t}$ *for all* $t \in [0, T - t_0)$ *in the domain of existence of* $\Sigma_t^-$. *Second, for any smooth, strictly convex surface* $\Sigma_0^+$ *which encloses* $\Sigma_{t_0}$ *for some time* $t \in (0, T)$, *the surfaces* $\Sigma_t^+$ *given by solving Equation 1.1 contain in* $\Sigma_{t_0+t}$ *for all* $t \in [0, T - t_0)$.

## 5.2 Comparison principle

**Proposition 37. (Comparison principle)** *: Let* $\Sigma_0$ *be a surface of class* $C_s^{2+\alpha,p}$ *such that is satisfies condition I, and let* $\Sigma^+$ *be a smooth, strictly convex surface containing* $\Sigma_0$ *at time* $t = 0$, *then the surface* $\Sigma_t$ *obtained by evolving* $\Sigma_0$ *by the Equation 1.1, is contained in the surface* $\Sigma_t^+$ *obtained by evolving* $\Sigma_0^+$ *by the Equation 1.1 up to the time of existence of* $\Sigma_t$. *Analogously, if* $\Sigma_0$ *contains a smooth, strictly convex surface* $\Sigma^-$ *at time* $t = 0$, *then the surface* $\Sigma_t$ *contains the surface* $\Sigma_t^-$ *obtained by evolving* $\Sigma_0^-$ *by the Equation 1.1 up to the time of existence of* $\Sigma_t$.

*Proof.* At first we observe that by the Maximum Principle the surfaces $\Sigma_t$ and $\Sigma_t^+$ cannot touch were they are both strictly convex. Suppose there exists a time $\bar{t}$ where they first touch at a point $\bar{P}$, then this obviously cannot happen in the interior of the flat side, thus $\bar{P}$ has to belong to the boundary of the flat side. Suppose $\Sigma_t$ has the flat side on the $x = 0$ plane, then the tangent to the surface at the point $\bar{P}$ would be parallel to $x = 0$ plane. This is because of the particular shape of the surface. Now if the two regions touched, because they are of class $C^1$ we would have that the tangent to $\Sigma_{\bar{t}}^+$ at $\bar{P}$ would be parallel to the $x = 0$ plane. But $\Sigma_{\bar{t}}^{+}$ is strictly convex, hence this would imply that a part of $\Sigma_{\bar{t}}^{+}$ is inside $\Sigma_{\bar{t}}$. This leads to a contradiction, and, therefore, the two regions never touch. The second part of the proof is straightforward since if $\Sigma_0$ contains a smooth, strictly convex surface $\Sigma^{-}$ at time $t = 0$, then the two surfaces cannot touch at the flat side of $\Sigma_t$ because the flat side does not move in its normal direction. Once again, by the Maximum Principle for parabolic equations the two surfaces cannot touch where they are strictly convex either. This observation concludes the proof of the proposition. $\square$

As a corollary we have a nice characterization of our solution of the *Harmonic Mean Curvature Flow.*

## 5.3 Viscosity solutions

**Corollary 38. (Viscosity solutions)** *Given an initial convex $\Sigma$ surface of class $C^{1,\alpha}$ with a flat side as in Theorem 13, then the solution of the HMCF $\Sigma_t$ found in Theorem 13 is a viscosity solution which converges uniformly to the initial surface $\Sigma$ as $t \to 0$. Moreover, this solution is unique.*

*Proof.* It is a consequence of the comparison principle. We show that our solution is a viscosity solution. We check that the first condition of viscosity solution, as defined at the beginning of this chapter, is satisfied: If $\Sigma_0^{-}$ is contained within $\Sigma_0$, then by the comparison principle $\Sigma_t^{-}$ is enclosed by $\Sigma_t$ for $t > 0$. The second condition also follows easily: Any surface $\Sigma_0^{+}$ which encloses $\Sigma_0$, and so by comparison principle $\Sigma_t^+$ encloses $\Sigma_t$ for $t > 0$.

*Uniqueneness.* Suppose we have two solutions $\{\Sigma_t^1\}$ and $\{\Sigma_t^2\}$ of the Main Theorem 13. Then, we show that these solutions coincide. We remind that the surfaces $\{\Sigma_t^1\}$ and $\{\Sigma_t^2\}$ are both given by the union of a strictly convex surface and a flat surface. Because they satisfy the parabolic Equation (2.1) and the non-degeneracy condition I, both flat sides evolve by the Curve Shortening Flow as shown in Chapter III. By the Maximum Principle, it follows that since the two surfaces $\{\Sigma_t^1\}$ and $\{\Sigma_t^2\}$ are equal at the initial time $t = 0$ and they share the same flat side, they are the same surface.

□

## 5.4 Constant rate of decrease for the area

Finally we observe the following property of surfaces evolving by the *Harmonic Mean Curvature Flow.*

**Proposition 39. (Constant rate of decrease for the Area)** *Let $A$ be the surface area of a smooth, closed, strictly convex surface $\Sigma$ evolving by HMCF, then:*

$$\frac{dA}{dt} = -4\pi \tag{5.1}$$

*Proof.* For a strictly convex surface evolving with a certain speed $v$ in the direction of the inward normal vector $\vec{N}$ by the equation $\frac{dP}{dt} = v \ \vec{N}$ the area decreases by the First Variation Formula. This means that:

$$\frac{dA}{dt} = -\int_{\Sigma} v\, H dp \tag{5.2}$$

$H$ indicates the mean curvature as in 1.1. For more details about how to derive this general formula see [7]. Thus, for the *HMCF*, $v = K/H$, which implies by the Gauss-Bonnet formula that the area shrinks at a constant rate:

$$\frac{dA}{dt} = -\int_{\Sigma} v\, H dp = -\int_{\Sigma} K dp = -4\pi$$

$\square$

**Proposition 40.** *Let $\Sigma$ be a surface of class $C_s^{1+\alpha,p}$ satisfying the non-degeneracy condition I, then its surface area $A$ decreases at constant rate $4\,\pi$:*

*Proof.* For the above mentioned First Variation Formula, we have that the surface area $A_c$ of $\Sigma_2$, the strictly convex part of $\Sigma$ evolves by the formula:

$$\frac{dA_c}{dt} = -\int_{\Sigma_2} v\, H dp = \int_{\Sigma_2} K dp = -2\pi \tag{5.3}$$

Moreover, $\Sigma_1$, the flat side of $\Sigma$, evolves by the Curve Shrinking Flow, as a consequence, its surface area $A_t$ shrinks with rate $2\,\pi$. Hence, the surface area $A$ of the solution of the *HMCF* shrinks with speed $-4\,\pi$.

$\square$

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