Exploring the Brachistochrone Problem

LaDawn Haws and Terry Kiser

1. MAKING THE BRACHISTOCHRONE ACCESSIBLE. In light of the attention given to a national crisis in mathematics education, concerned mathematics instructors are always looking for innovative ways to present and reinforce ideas. For a generation that grew up with fast paced MTV and special effects movies like Star Wars, the classroom may appear to be a fairly dull environment with uncompromising standards. Computer technology can help educators compete for students' attention and at the same time enhance the learning process by

1) bringing an added dimension—visualization—to the presentation of mathematical concepts,
2) giving students greater flexibility to explore and discover ideas on their own,
3) making more advanced topics accessible to a wider range of classes.

These learning aspects will be discussed in the context of some *Mathematica* packages for exploring the classic Brachistochrone problem and interesting variations.

The Brachistochrone Problem, to find the curve joining two points along which a frictionless bead will descend in minimal time, is typically introduced in an advanced course on the Calculus of Variations. The statement of this problem is easily understood, even for high school students, when phrased in a more familiar context as follows: "What shape should a roller coaster track have so the car will travel from a high point A to a low point B as fast as possible?" This form of the statement of the problem, however, has resulted in some unexpected and amusing responses from students who were asked to draw what they thought would be a "fast track."

The authors have written a *Mathematica* command called **Race** that allows students to explore this problem graphically. We have developed several activities and exercises for students with a wide variety of mathematical abilities, from algebra to differential equations. The student may design a path, or several paths, and **Race** will produce a plot of the paths, their lengths, times of descent and, optionally, an animation of beads racing down the paths.

The exciting part is that **Race** enables students to experiment on their own with different shaped curves to gain intuition and formulate criteria for a "fast curve" without needing the mathematical expertise to solve the problem. Articulating criteria such as,

- the curve should start out with a steep descent to build up velocity quickly, but
- the steep part should not be "too long" or the advantage gained in increased acceleration will be lost in increased path length,

requires a good understanding of slope, velocity, acceleration, and arc-length; fairly sophisticated stuff for pre-calculus or even high school students!

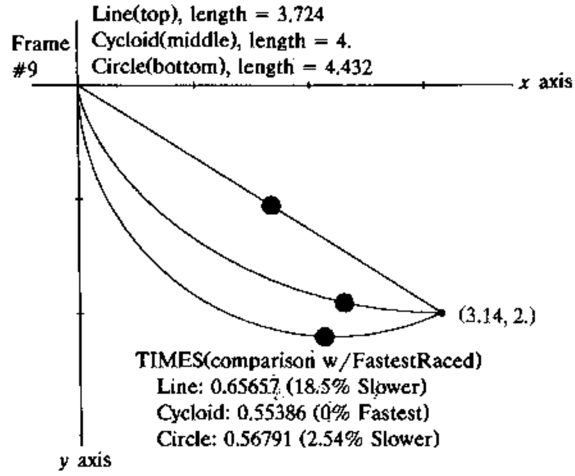

Figure 1. A *Mathematica* simulation generated by **Race**

The well known solution curve, the cycloid, makes the optimal compromise between steepness versus path length and is easily derived from the Euler-Lagrange equation. Of course, this tool is not available in most undergraduate mathematics classes, but that does not mean the underlying problem is inaccessible to these students. Instead of simply presenting the cycloid as an interesting example of a parametric curve, as is typically done in a 1st or 2nd semester calculus class, its special features can be developed—in fact, the students can discover them for themselves.

Students in a beginning differential equations course can understand the derivation and solution of the differential equation governing the Brachistochrone. They are already familiar with minimization criteria (to minimize $f(x)$, consider solutions to $f'(x) = 0$), so the Euler-Lagrange equation is not hard for them to swallow and should not be a deterrent from investigating this application.

Use of *Mathematica* as a basis for exploring the Brachistochrone problem is a prime example of how technology can allow students to go beyond standard textbook applications and address more realistic or current applications. For example, a natural question to ask concerning the Brachistochrone problem is what happens if friction or air resistance is included in the model? This is discussed later in this article. The messy calculations that are typical in many real-world applications and (up to now) made them off limits in the undergraduate classroom can be handled by the computer, with possibly some surprising results to student and instructor alike.

There are many applications that computer technology now makes accessible at all levels. One challenge to all of us as educators is to make creative use of this technology. The *Mathematica* command **Race** and accompanying packages along with a notebook of examples is available upon request. This article will conclude with some specific examples illustrating these exercises.

2. PRE-CALCULUS AND CALCULUS. An engaging way to present this topic is to begin by testing the students' intuition. We have broken classes up into groups and asked them to draw and discuss what they think is the fastest curve. In addition, we have created a *Mathematica* package that allows students to "draw" their curve on the computer screen and generates a simulation from a speedier, scaled down version of **Race** (this has been used several times to provide a computer lab experience for 7th–9th grade students as a part of a "Math Field Day" project held annually at Chico State University). We next give a "live" demonstration of marbles racing down wooden ramps, roughly in the shape of a cycloid and a straight line. This demonstration generates a great deal of excitement, which just goes to show that the "old technology" still has its place. It also provides a concrete time scale which is necessary for qualifying what constitutes a "close" race. Even in our differential equations classes where the majority of the students deduce on their own that the marble rolling down the cycloidal ramp will beat the marble on the straight line ramp, many are surprised at the margin of victory.

The focus of the presentation should be a graphical exploration of the criteria that makes the cycloid a fast curve, not on its derivation as the fastest curve. The details concerning the cycloid can be adjusted depending on the level of the class. We use a *Mathematica* animation to present the cycloid as the curve generated by tracing a point on a rolling circle (due to Stan Wagon at Macalester College) and in a Calculus course it is appropriate to derive the parametric equations. Even after the cycloid is presented as the solution to the Brachistochrone problem, there are many interesting questions to investigate that depend on physical or graphical intuition. For example, a traditional graphing exercise for a pre-calculus class can be spiced up by asking for the fastest curve among a class of familiar functions (especially appropriate if these functions have recently been studied) but with an unknown parameter. This motivates the need to graph several examples of the function in question so they can apply their newly gained intuition on what makes a fast curve. Then, they can **Race** their graphs to check their intuition. Test your intuition on the examples below.

2.1. Finding a Fast Parabola. Find the fastest parabola that starts at the origin and ends at a given point, say, $(3, 2)$ (in this article we will always take the positive y-axis to be oriented downward). Since there are three unknowns in a quadratic, we are free to impose one more condition. Let's take the x-coordinate of the vertex, m, to be our unknown parameter. Below are graphs for $m = 2$, 2.5, 3, and 3.5. Which of these is the fastest curve and how does this value of m compare with the optimal value for m giving the fastest of all such parabolas? The cycloid ending at the point $(3, 2)$ is included to add some perspective. The *Mathematica* generated plot of times of descent versus m for a wide range of values of m indicates that the fastest of all such parabolas is found with $m \approx 2.5$ (not the parabola with its vertex at the ending point which is a popular choice—numerical minimization verifies that $m \approx 2.494$).

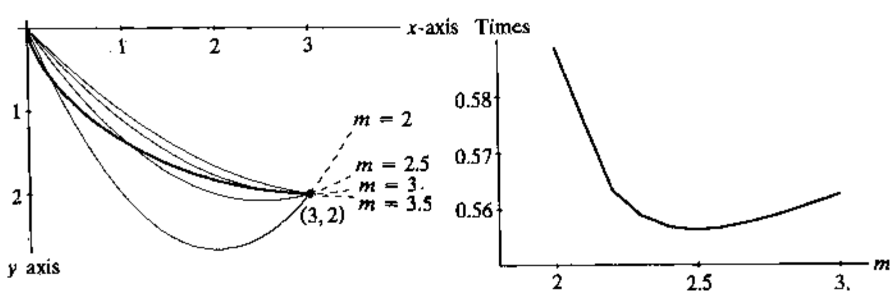

Figure 2. Finding a fast parabola

2.2. Finding the Fastest *N*th Root. This is a good family of curves for exploring the trade off between steepness versus path length. Below are graphs of $y = 2(x/\pi)^{1/n}$, for $n = 2$, 4, and 6. The fastest of these curves is very competitive with the cycloid, being only slightly more than 1% slower; this would not be discernible with a live model. Can you tell which one it is?

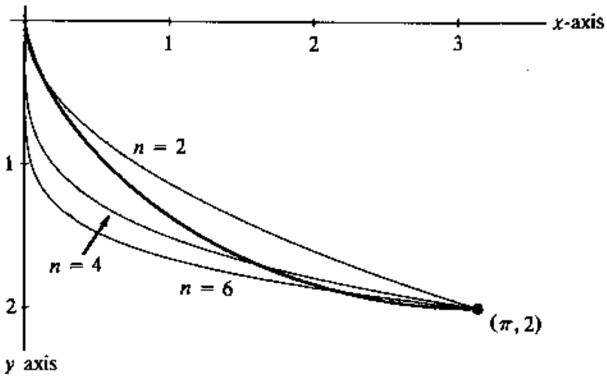

Figure 3. *N*th roots-Steep vs Path length

The answer is $n = 2$. This has always surprised our audiences, both faculty and students, especially when a plot of the cycloid is not included.

3. ADDING A SMALL DOSE OF REALITY. Inevitably, any discussion about the importance of computer technology to mathematics will bring up the ability to address more realistic applications. We strongly agree and yet here we are presenting an application that is only valid if it takes place in a vacuum and we ignore frictional forces! It is vital that students don't leave our classes with incorrect insight because of the setting we choose to present an application. Is this a critical issue for this problem? What happens if kinetic friction or air resistance is included in the Brachistochrone model? The *Mathematica* simulation allows the user to include a coefficient of friction to see its effect on the descent time for any curve. A typical coefficient of friction, to be denoted as μ, will be less than or equal to 0.1. A cycloid ending at the point $(\pi, 2)$ with $\mu = .1$ is approximately 3% slower than a cycloid with no friction.

Although kinetic friction has seemingly little effect on the travel time, the question still remains as to what is the effect on the shape of the fastest curve when friction is included in the model for the Brachistochrone problem. Is the cycloid still the fastest curve? If there is a new fastest curve, is it initially steeper than the cycloid lying below it or shallower and lying above it? Physical insight alone can play a key role in answering these qualitative questions. The solution when kinetic friction is included (this will be derived below for a simplified model of friction) has significantly different graphical characteristics and is an interesting generalization of the cycloid.

It is helpful to begin by motivating what graphical and physical insight suggests will be the effect of including friction. Depending on the level of the course, the solution can be derived or merely presented graphically and compared to other curves. The frictional force is assumed to be proportional to the normal component of the weight of the bead and acting in the negative tangential direction (see Figure 4). Due to the curvature of the path, the normal component of acceleration

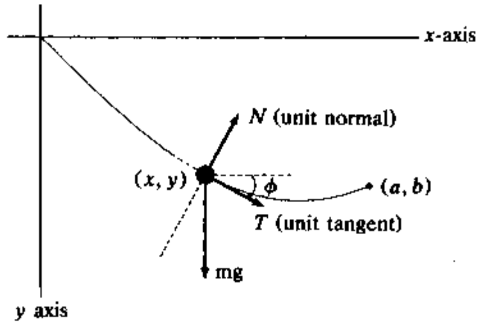

Figure 4

also contributes to the frictional force. We generally neglect this component of the friction in our initial discussions and derivations with students in a differential equations course. Students are more familiar with the weight component from studying inclined planes in physics and this is usually challenge enough. The more realistic solution, however, is presented graphically and can lead to interesting discussions as to why it differs from the cycloid or the new Brachistochrone using the simpler model of friction. This simplified model incorporates some interesting qualitative changes and has the additional advantage that the derivation of the solution is accessible in an introductory differential equations course since the equation of motion remains integrable (a more accurate treatment requires a constrained variational technique and, amazingly enough, it can still be solved in terms of elementary, albeit, messy functions—see [1]).

Before proceeding with a derivation of the solution, let's develop some qualitative insight by doing a simple physical analysis, that is, let's compare the forces with friction included versus no friction where we know the shape of the solution is a cycloid. Neglecting curvature, the magnitude of the force of friction is less at steep points on a curve, ranging from zero at a vertical tangent to the whole weight at a horizontal tangent. Since the lesson learned from the classical Brachistochrone problem, heuristically speaking, is that steepness is most important initially, this suggests that steepness will now be given more weight (versus path length) and the optimal curve, which should still have an initial vertical tangent, will be slightly steeper or below the cycloid (at least for the "beginning" portion of the curve). Since the normal component of acceleration is proportional to the square of the speed, one might expect just the opposite to be the case when it is included in the model for friction. Starting off steeper would force more curvature for the latter portion of the path when there is a greater velocity.

3.1. Derivation of the Fastest Curve With Kinetic Friction. Take the starting point to be the origin and orient the positive y-axis downward. We seek the fastest curve $y(x)$ starting at $(0,0)$ and ending at an arbitrary point (a,b).

If we ignore friction, then we can apply the conservation of energy, or equivalently, equate work with change in kinetic energy to obtain $v = \sqrt{2gy}$ where the velocity v is given as ds/dt. Including friction forces us to do a line integral to find the work or, alternatively, we can start with the equation of motion as follows. At a point (x, y) on the curve, the unit tangent and normal vectors, illustrated in Figure

4, can be written in terms of arc-length s as,

$$\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \quad \text{and} \quad \mathbf{N} = -\frac{dy}{ds}\mathbf{i} + \frac{dx}{ds}\mathbf{j}.$$

The forces of gravity and friction are given by,

$$\mathbf{F}_{gravity} = mg\,\mathbf{j} \quad \text{and} \quad \mathbf{F}_{friction} = -\mu(\mathbf{F}_{\text{gravity}} \cdot \mathbf{N})\mathbf{T} = -\mu mg\frac{dx}{ds}\mathbf{T}.$$

So, the components along the curve (i.e. in the direction of $\mathbf{T}$) are

$$\mathbf{F}_{gravity} \cdot \mathbf{T} = mg\frac{dy}{ds} \quad \text{and} \quad \mathbf{F}_{friction} \cdot \mathbf{T} = -\mu mg\frac{dx}{ds}.$$

Using these components in Newton's first law gives,

$$m\frac{dv}{dt} = mg\frac{dy}{ds} - \mu mg\frac{dx}{ds} \tag{1}$$

and substituting

$$\frac{dv}{dt} = v\frac{dv}{ds} = \frac{1}{2}\frac{d}{ds}(v^2)$$

into (1) yields, after integration w.r.t. s,

$$\frac{1}{2}v^2 = g(y - \mu x) \quad \text{or} \quad v = \sqrt{2g(y-\mu x)}\,.$$

Apply the chain rule to $v = ds/dt$ and use the arc-length formula for ds/dx to solve for dx/dt as a function of x, which can be inverted to give the total time,

$$\mathbf{T}(x, y, y') = \int_a^b \sqrt{\frac{1 + (y')^2}{2g(y-\mu x)}}\,dx\,. \tag{2}$$

Since the computations become quite messy, what follows will just be an outline of the major steps. Apply the Euler-Lagrange equation,

$$\frac{d}{dx}(F_{y'}) - F_y = 0$$

where F is the integrand in equation (2) to obtain the 2nd order differential equation,

$$\left(1 + (y')^2\right)(1 + \mu y') + 2(y - \mu x)y'' = 0.$$

Through two substitutions and a partial fractions integration, this can be reduced to,

$$\frac{1 + (y')^2}{(1 + \mu y')^2} = \frac{C}{y - \mu x}, \tag{3}$$

for some non-negative constant C.

Following the lead from the classical problem, the substitution $y' = \cot(\theta/2)$ into (3) can be used to obtain a parametric solution for the optimal curve. Denoting the parametrization for the cycloid as,

$$x_c(\theta) = \rho(\theta - \sin\theta) \quad \text{and} \quad y_c(\theta) = \rho(1 - \cos\theta),$$

the new fastest curve for the "frictional" Brachistochrone problem can be given in the form,

$$x(\theta) = x_c(\theta) + \mu\rho(1 - \cos\theta)$$
$$y(\theta) = y_c(\theta) + \mu\rho(\theta + \sin\theta). \tag{4}$$

The parameterizations in (4) and for the cycloid are valid for a range $0 \le \theta \le \theta_f$, where ρ and θ_f must be determined so the curves pass through the ending point (a, b).

Figure 5 compares this new curve with the cycloid. Note the similar repetitive pattern with vertical tangents at even multiples of $\pi\rho$; however, the minimums do not occur at the same place. We have indicated a sloping line at which this new curve stops as opposed to making it back to the x-axis.

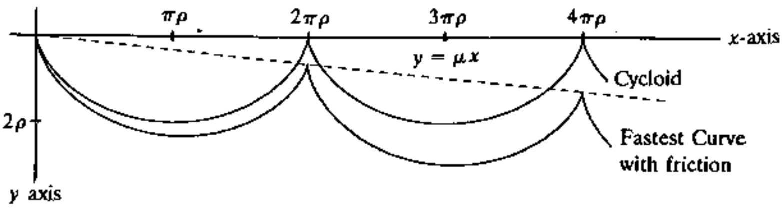

Figure 5. A generalization of the Cycloid

This line is called the "line of repose" and has the minimum slope, μ, for which the bead will begin to slide. The solution derived above is not valid for $y < \mu x$. This places a restriction on allowable ending points, which is consistent with our physical insight that, due to the loss of energy to friction, the bead can't make it back to its original height. Given a valid ending point, there is a unique curve of this form starting at the origin with a vertical tangent. Figure 6, generated from **Race**, gives a better comparison of this "frictional" Brachistochrone versus the classic Brachistochrone, a cycloid. Both were raced with a coefficient of friction $\mu = 0.1$ and their times are compared to the cycloid without friction.

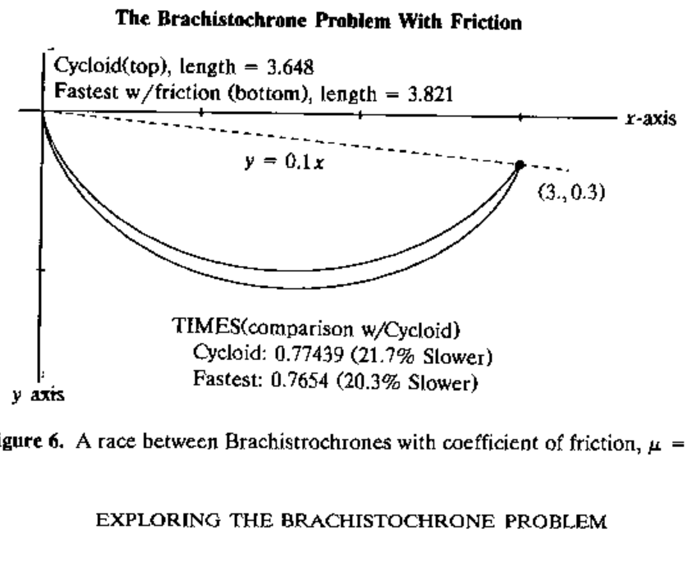

Figure 6. A race between Brachistrochrones with coefficient of friction, $\mu = 0.1$

For all valid ending points, the new fastest curve lies below the cycloid throughout its entire length, and placing the ending point on the line of repose gives the greatest distinction between the two curves. With a more accurate treatment of the frictional forces, as mentioned earlier, the fastest curve lies *above* the cycloid for its entire length (see Figure 7 with $\mu = 0.1$) and the ending point must satisfy the *strict* inequality, $y < \mu x$, since the bead takes infinite time to reach the line of repose. Consistent with our earlier intuition that the normal component of acceleration puts a higher penalty on curvature, Figure 7 indicates that as the coefficient of friction increases the more realistic frictional Brachistochrone will approach a straight line.

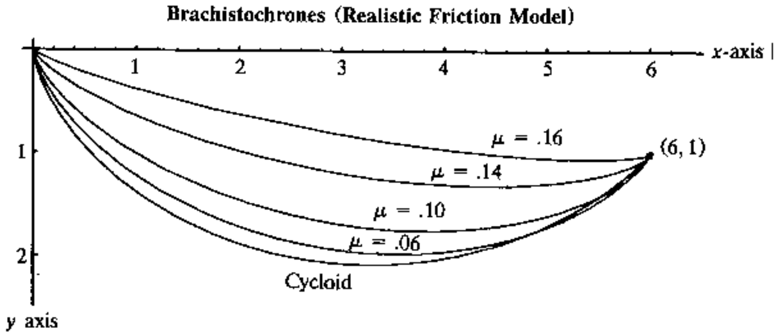

Figure 7. Realistic "frictional" Brachistochrones approaching a straight line

The reader should not be misled into thinking that all of the graphical and physical intuition that has been developed in this article was known to the authors before beginning a computer-aided investigation of the Brachistochrone problem and variations. In fact, this is one of the main points of this article; just as the authors gained valuable insight through computer generated graphics and simulations, so will students at all levels increase their mathematical intuition with these tools and be provided the opportunity to begin exploring questions they would have never even thought of before.

4. ADDITIONAL PROBLEMS. We have looked at other variations of the classic Brachistochrone problem. Many interesting questions can be generated by restricting the class of admissible curves, as with the fastest parabola or nth root problems given earlier. One that we find particularly enjoyable is the "Two Line" Brachistochrone problem: Find the "break" point for the fastest two straight line segment curve joining the origin and the ending point. A similar problem has appeared several times in the problems section of the MAA Monthly; there the question was whether or not the time of travel along the two line segments were equal for the optimal break point (answered in E1255 [1977, 652]). A program like *Mathematica* makes it easy for students to find the break point numerically, which frees them to investigate other questions, such as the problem posed in the Monthly or to look for cases where there is a simple algorithm for finding the break point.

Similar to the constrained variational problem with friction included in the model, is the question of what effect air resistance has on the fastest curve. We are currently attempting to solve this problem numerically so the solution can be presented graphically in our classes and compared with other curves, although it seems likely that the effect of air resistance may be far less significant than the effect of friction.

The Two Lines problem is included in our Brachistochrone notebook and an article concerning the use of *Mathematica* to investigate more realistic treatments of friction and air resistance is in preparation. We invite questions or discussions on these problems and will send completed or partially completed materials upon request.

REFERENCE

1. N. Ashby, W. E. Brittin, W. F. Love and W. Wyss, Brachistochrone With Coulomb Friction, *American Journal of Physics*, Vol. 43, No. 10 (October 1975), 902–905.

Department of Mathematics & Statistics
California State University, Chico
Chico, CA 95929
lhaws@oavax.csuchico.edu
tkiser@oavax.csuchico.edu

PICTURE PUZZLE
(from the collection of Paul Halmos)

What conspiracy is this?
(see page 344.)