Indifference valuation and risk monitoring of unhedgeable risks

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References

- "A valuation algorithm for indifference prices in incomplete markets",

  Finance and Stochastics (2004)

- "An example of indifference prices under exponential preferences",

  Finance and Stochastics (2004)

- "Derivative pricing, investment management and the term structure of exponential utilities: the case of binomial model"

  Indifference Pricing, PUP (2005)

Pricing elements for contingent claims

- Arbitrage free valuation theory
- Actuarial principles of premium calculations
- Pricing via investment optimality

Actuarial principles of premium calculation

Statistical Risks

- Probability space

$$\Omega = \{\omega_1, \omega_2\}, \quad \mathbb{P}\{\omega_1\} = p, \quad \mathbb{P}\{\omega_2\} = 1 - p$$

- Risk: $Y(\omega) \quad Y(\omega_1) = Y^u, \quad Y(\omega_2) = Y^d$

Premium Calculation

- Expected value: $\nu = E_{\mathbb{P}}(Y)$

- Standard deviation: $\nu = E_{\mathbb{P}}(Y) + a\sqrt{\mathrm{Var}_{\mathbb{P}}(Y)}, \ a > 0$

- Variance: $\nu = E_{\mathbb{P}}(Y) + b\,\mathrm{Var}_{\mathbb{P}}(Y), \ b > 0$

Premium calculation based on risk preferences

- Indifference (x—initial wealth)

$$E_{\mathbb{P}}(U(x+\nu-Y)) = U(x)$$

- Zero utility (corresponds to $x=0$)

$$E_{\mathbb{P}}(U(\nu-Y)) = U(0)$$

- Certainty equivalent

$$\nu = -U^{-1}(E_{\mathbb{P}}U(-Y))$$

Exponential Utility

$$U(x) = -e^{-\gamma x}, \qquad x \in R,\ \gamma > 0$$

Indifference $\longleftrightarrow$ **zero utility** $\longleftrightarrow$ **certainty equivalent**

$$\nu(Y) = -\frac{1}{\gamma}\log E_{\mathbb{P}}(e^{-\gamma Y}) = E_{\mathbb{P}}(Y) + \frac{1}{2}\gamma\,\mathrm{Var}_{\mathbb{P}}(Y) + o(\gamma)$$

Arbitrage free valuation theory

Hedgeable risks

- Riskless bond B (zero interest rate): $B_0 = B_T = 1$
- Probability space: $\Omega\{\omega_1, \omega_2\}$, $\mathbb{P}(\omega_1) = p$, $\mathbb{P}(\omega_2) = 1 - p$
- Traded asset S: $S = S^d, S^u$ with $0 < S^d < S^u$
- Claim C : $C_T = C(S_T)$

Pricing by arbitrage

- Arbitrage free measure $\mathbb{Q}$: $\mathbb{Q}\{\omega_1\} = q = \frac{S_0 - S^d}{S^u - S^d}$, $\mathbb{Q}\{\omega_2\} = 1 - q$

Arbitrage free price $\nu(C_T)$

$$\boxed{\nu(C_T) = E_{\mathbb{Q}}(C(S_T))}$$

Incomplete Models

- Probability space

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, \quad \mathbb{P}\{\omega_i\} = p_i, \quad i = 1, ..., 4$$

- Two risks

- Random variables S_T and Y_T

$$S_T(\omega_1) = S^u, Y_T(\omega_1) = Y^u \qquad S_T(\omega_3) = S^d, Y_T(\omega_3) = Y^u$$

$$S_T(\omega_2) = S^u, Y_T(\omega_2) = Y^d \qquad S_T(\omega_4) = S^d, Y_T(\omega_4) = Y^d$$

Issues

- There are hedgeable and unhedgeable risks to specify
- The hedgeable risks could be priced by arbitrage, and the hedgeable ones by certainty equivalent
- Two probability measures, the nested risk neutral measure and the historical measure, are then involved

Arbitrary risks: $C_T = C(Y_T, S_T)$

Pricing from a perspective of optimal investment

- Use the market to assess both types of risk
- Formulate optimality criteria in terms of individual preferences
- Derive the concept of value from indifference to the various investment opportunities

Fundamental elements of an indifference pricing system

- Monotonicity, scaling and concavity with respect to payoffs
- Monotonicity, robustness and regularity with respect to risk aversion
- Consistency with the no-arbitrage principle
- Translation invariance with respect to hedgeable risks
- Risk quantification and monitoring
- Numeraire independence
- Additivity with respect to incremental risks
- Risk transfering across parties

Investment opportunities

- We invest the amount β in bond and the amount α in stock
- Wealth variable

$$X_0 = x, \quad X_T = \beta + \alpha S_T = x + \alpha(S_T - S_0)$$

Indifference price

- For a general claim C_T , we define the value function

$$V^{C_T}(x) = \max_{\alpha} E(-e^{-\gamma(X_T - C_T)})$$

- The indifference price is the amount $\nu(C_T)$ for which,

$$\boxed{V^0(x) = V^{C_T}(x + \nu(C_T))}$$

Structural result

Duality techniques yield

$$\nu = \sup_Q (E_Q(C_T) - \vartheta(C_T)),$$

$$\vartheta(C_T) = \frac{1}{\gamma} H(Q/P) - \inf_Q \left(\frac{1}{\gamma} H(Q/P) \right)$$

Rouge and El Karoui (2000)

Kabanov and Stricker (2002)

Frittelli (2000)

Delbean et al. (2000)

Considerations

- Price respresented via a non-intuitive optimization problem
- Pricing measure depends on the payoff
- Certain pricing elements are lost

A probabilistic algorithm for indifference prices

- **Arbitrage free prices** $\boxed{\nu(C_T) = E_{\mathbb{Q}^*}(C_T)}$

$E(.)$: linear pricing functional
$\mathbb{Q}^*$: the (unique) risk neutral martingale measure

- **Indifference prices** $\boxed{\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T)}$

$\mathcal{E}$: pricing functional
(possibly) nonlinear
payoff independent
wealth independent
preference dependent

$\mathbb{Q}$: pricing measure
payoff independent
preference independent

The indifference price

$$\nu(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log(E_{\mathbb{Q}}(e^{\gamma C(S_T,Y_T)} \mid S_T))\right) = \mathcal{E}_{\mathbb{Q}}(C_T)$$

$$\mathbb{Q}(Y_T \mid S_T) = \mathbb{P}(Y_T \mid S_T)$$

Examples

- $C(S_T, Y_T) = c_1(S_T) + c_2(Y_T)$ and S, Y independent under $\mathbb{P}$

$$\nu(c_1(S_T) + c_2(Y_T)) = E_{\mathbb{Q}}(c_1(S_T)) + \frac{1}{\gamma}\log(E_{\mathbb{Q}}(e^{\gamma c_2(Y_T)}|S_T)$$

$$= E_{\mathbb{Q}}(c_1(S_T)) + \frac{1}{\gamma}\log E_{\mathbb{P}}(e^{\gamma c_2(Y_T)})$$

- $C(S_T, Y_T) = c_1(S_T) + c_2(Y_T)$ and Y is functionally dependent on S

$$\nu(c_1(S_T) + c_2(Y_T)) = E_{\mathbb{Q}}(c_1(S_T) + c_2(Y_T))$$

- $C(S_T, Y_T) = c_1(S_T) + c_2(S_T, Y_T)$ with Y and S, in general, correlated

$$\nu(c_1(S_T) + c_2(S_T, Y_T))$$

$$= E_{\mathbb{Q}}(c_1(S_T)) + \frac{1}{\gamma}\log(E_{\mathbb{Q}}(e^{\gamma c_2(S_T, Y_T)} \mid S_T))$$

$$= E_{\mathbb{Q}}(c_1(S_T)) + \mathcal{E}_{\mathbb{Q}}(c_2(S_T, Y_T))$$

The pricing functional is translation invariant with repect to hedgeable risks

Valuation Procedure

$$\nu(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log(E_{\mathbb{Q}}(e^{\gamma C(S_T,Y_T)} \mid S_T)\right) = \mathcal{E}_{\mathbb{Q}}(C_T)$$

$$\mathbb{Q}(Y_T \mid S_T) = \mathbb{P}(Y_T \mid S_T)$$

- Step 1: **Specification, isolation and pricing of unhedgeable risks**

The original payoff C_T is altered to the preference adjusted payoff

$$\tilde{C}_T = \frac{1}{\gamma}\log\left(E_{\mathbb{Q}}(e^{\gamma C(S_T,Y_T)}|S_T)\right)$$

Observe that $\tilde{C}_T \neq \frac{1}{\gamma}\log(E_{\mathbb{Q}^*}(e^{\gamma C(S_T,Y_T)})) \neq \frac{1}{\gamma}\log(E_{\mathbb{P}}(e^{\gamma C(S_T,Y_T)}))$

but $\tilde{C}_T = \frac{1}{\gamma}\log(E_{\mathbb{Q}}(e^{\gamma C(S_T,Y_T)} \mid S_T)) = \frac{1}{\gamma}\log(E_{\mathbb{P}}(e^{\gamma C(S_T,Y_T)} \mid S_T))$

- Step 2: **Pricing by arbitrage of the remaining hedgeable risks**

Conclusions

$$\boxed{\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T)}$$

Pricing functional $\mathcal{E}_{\mathbb{Q}}(.)$

Nonlinear

Payoff independent

Preference dependent in the first (unhedgeable risk) step only

Pricing measure $\mathbb{Q}$

Preserves conditional distribution of unhedgeable risks, given the hedgeable ones

Preference independent

Payoff independent

Price properties

- Monotonicity, scaling and concavity w.r.t. payoffs

$\nu(0) = 0,$

$\nu(aC_T) \leq a\nu(C_T) \quad a \in (0,1)$

$\nu(aC_T) \geq a\nu(C_T) \;\; a \geq 1$

$\nu(aC_{1,T} + (1-a)C_{2,T}) \leq a\nu(C_{1,T}) + (1-a)\nu(C_{2,T}) \;\; a \in (0,1)$

- Monotonicity, robustness and regularity w.r.t. risk aversion

$$\gamma \to \nu(C_T;\gamma) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}(e^{\gamma C(S_T,Y_T)}|S_T)\right)$$

is increasing and continuous.

$$\nu(C_T;0^+) = E_{\mathbb{Q}}(C_T) \qquad\qquad \nu(C_T;\infty-) = E_{\mathbb{Q}}\|C_T\|_{L^\infty_{\mathbb{Q}\{.|S_T\}}}$$

$$\nu'(C_T; 0^+) = \frac{1}{2} E_{\mathbb{Q}}(\text{Var}(C_T \mid S_T))$$

$$\nu(C_T; \gamma) = E_{\mathbb{Q}}(C_T) + \frac{1}{2}\gamma E_{\mathbb{Q}}(\text{Var}(C_T \mid S_T)) + o(\gamma)$$

- Consistency with the no arbitrage principle

Let Q_e be the set of absolutely continuous with $\mathbb{P}$ martingale measures. Then, $\nu(C_T)$ belongs to the arbitrage free interval, namely,

$$\boxed{\inf_{Q \in Q_e} E_Q(C_T) \leq \mathcal{E}_{\mathbb{Q}}(C_T) \leq \sup_{Q \in Q_e} E_Q(C_T)}$$

- Translation invariance w.r.t. hedgeable risks

$$\nu(c_1(S_T) + c_2(S_T, Y_T)) = \mathcal{E}_{\mathbb{Q}}(c_1(S_T) + c_2(Y_T))$$

$$= E_{\mathbb{Q}}(c_1(S_T)) + \mathcal{E}_{\mathbb{Q}}(c_2(S_T, Y_T))$$

- The indifference price and convex risk measures

$$\rho(C_T) = \nu(-C_T)$$

$\rho(\cdot)$: capital requirement

$\nu(\cdot)$: outcome of optimal behavior

Risk quantification and monitoring
Hedgeable risks

- C_T: $\mathcal{F}_T^S$-measurable claim

$$C_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0)$$

with

$$\nu(C_T) = E_{\mathbb{Q}}(C_T) \quad \text{and} \quad \frac{\partial \nu(C_T)}{\partial S_0} = \frac{\partial E_{\mathbb{Q}}(C_T)}{\partial S_0}$$

Unhedgeable risks

- Optimal investments in the presence of the claim

$$\alpha^{C_T,*} = \alpha^{0,*} + \frac{\partial \nu(C_T)}{\partial S_0},$$

where

$$\alpha^{0,*} = -\frac{1}{\gamma}\frac{\partial H\left(\mathbb{Q}\,|\mathbb{P}\right)}{\partial S_0}$$

- Optimal wealths with and without the claim

$$X_T^{C_T,*} = x + \nu(C_T) + \alpha^{C_T,*}(S_T - S_0) \quad \text{and} \quad X_0^{C_T,*} = x + \nu(C_T)$$

$$X_T^{0,*} = x + \alpha^{0,*}(S_T - S_0) \quad \text{and} \quad X_T^{0,*} = x$$

- The optimal residual wealth

$$\boxed{L_t = X_t^{C_T,*} - X_t^{0,*} \quad \text{for} \quad t = 0, T}$$

Indifference hedge and replicability

$$L_0 = \nu(C_T)$$

and

$$L_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0)$$

Conditional certainty equivalent and extraction of unhedgeable risks

$$L_T = \tilde{C}_T$$

Martingale property

$$E_Q(L_T) = L_0 = \nu(C_T) \quad \text{for } Q \in \mathcal{Q}_e$$

- The residual risk R_t

$$\boxed{R_t = C_t - L_t \quad \text{for} \quad t = 0, T}$$

Nonreplicable components of the claim and of the residual risk

$$R_0 = 0 \quad \text{and} \quad R_T = C_T - \tilde{C}_T \; ; \qquad \tilde{R}_T = 0$$

The indifference price of the residual risk

$$\boxed{\nu(R_T) = 0 \qquad\qquad R_T \neq 0}$$

Supermartingale property under the pricing measure $\mathbb{Q}$

$$E_{\mathbb{Q}}(R_T) \leq R_t = 0$$

Actuarial certainty equivalent

$$\frac{1}{\gamma} \log E_{\mathbb{P}}(e^{\gamma R_T}) = 0$$

Residual risk decomposition

The supermartingale R_t, for $t = 0, T$, admits the decomposition

$$R_t = R_t^m + R_t^d$$

where

$$R_0^m = 0 \quad \text{and} \quad R_T^m = R_T - E_{\mathbb{Q}}(R_T),$$

and

$$R_0^d = 0 \quad \text{and} \quad R_T^d = E_{\mathbb{Q}}(R_T).$$

The component R_t^m is an $\mathcal{F}_T^{(S,Y)}$-martingale under $\mathbb{Q}$ while R_t^d is decreasing and adapted to the trivial filtration $\mathcal{F}_0^{(S,Y)}$.

Error quantification

The expected residual risk satisfies

$$E_{\mathbb{Q}}(R_T) = -\frac{1}{2}\gamma E_{\mathbb{Q}}\Big(Var_{\mathbb{Q}}\left(C_T\,|S_T\right)\Big) + o\left(\gamma\right)$$

and

$$E_{\mathbb{Q}}(R_T) = -\frac{1}{2}\gamma E_{\mathbb{Q}}\Big(Var_{\mathbb{Q}}\left(R_T\,|S_T\right)\Big) + o\left(\gamma\right)$$

Payoff decomposition

Let $\tilde{C}_T$ and R_T be, respectively, the conditional certainty equivalent and the residual risk associated with the claim C_T. Let also R_t^m and R_t^d be the Doob decomposition components of the residual risk supermartingale.

Define the process $M_t^{\tilde{C}}$, for $t = 0, T$, by

$$M_0^{\tilde{C}} = \nu\left(C_T\right) \quad \text{and} \quad M_T^{\tilde{C}} = \nu\left(C_T\right) + \frac{\partial \nu\left(C_T\right)}{\partial S_0}(S_T - S_0)$$

The claim C_T admits the unique, under $\mathbb{Q}$, payoff decomposition

$$C_T = \tilde{C}_T + R_T$$

$$= \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0) + R_T = M_T^{\tilde{C}} + R_T^m + R_T^d$$

Indifference price decomposition

The indifference price process ν_t, for $t = 0, T$, defined by

$$\nu_0 = \nu(C_T) \quad \text{and} \quad \nu_T = C_T$$

is an $\mathcal{F}_T^{(S,Y)}-$ supermartingale under $\mathbb{Q}$.

It admits the unique Doob decomposition

$$\boxed{\nu_t = M_t + R_t^d = M_t^{\tilde{C}} + R_t^m + R_t^d}$$

The components M_t and R_t^d represent, respectively, the associated martingale and the non-increasing parts of the price process ν_t.

Relative indifference prices

Nonlinearity of indifference prices

$$\nu(\alpha C_T) \neq \alpha \nu(C_T) \quad \text{for} \quad \alpha \neq 0, 1$$

$$\nu(C_T^1 + C_T^2) \neq \nu(C_T^1) + \nu(C_T^2)$$

$$\nu(\sum_{i=1}^{N} C_T^i) \neq \sum_{i=1}^{N} \nu(C_T^i)$$

Quantification of incremental risks in reference to existing unhedgeable risk exposure

Stoikov (2004)

Relative indifference prices

Let ${C^1}_T = C^1(S_T, Y_T)$ and ${C^2}_T = C^2(S_T, Y_T)$ be two claims that have indifference prices $\nu(C_T^1)$ and $\nu(C_T^2)$. Let V^{C^1}, V^{C^2} and $V^{C^1+C^2}$ be the value functions corresponding to claims C_T^1, C_T^2 and $C_T^1 + C_T^2$.

The conditional indifference prices $\nu\left(C_T^2/C_T^1\right)$ and $\nu\left(C_T^1/C_T^2\right)$ are defined, respectively, as the amounts satisfying, for all wealth levels,

$$V^{C_1}(x) = V^{C_1+C_2}\left(x + \nu\left(C_T^2/C_T^1\right)\right)$$

and

$$V^{C_2}(x) = V^{C_1+C_2}\left(x + \nu\left(C_T^1/C_T^2\right)\right)$$

Linearity properties of relative indifference prices

Let ${C^1}_T = C^1(S_T, Y_T)$ and ${C^2}_T = C^2(S_T, Y_T)$ having indifference prices $\nu(C_T^1)$

and $\nu(C_T^2)$ and conditional indifference prices $\nu\left(C_T^1/C_T^2\right)$ and $\nu\left(C_T^2/C_T^1\right)$

$$\nu\left(C_T\right) = \nu\left(C_T^1\right) + \nu\left(C_T^2/C_T^1\right)$$

$$\nu\left(C_T\right) = \nu\left(C_T^2\right) + \nu\left(C_T^1/C_T^2\right)$$

Relative indifference prices

Price spreads and isommetries

$$\nu\left(C_T^1\right) - \nu\left(C_T^2\right) = \nu\left(C_T^1/C_T^2\right) - \nu\left(C_T^2/C_T^1\right)$$

Errors

$$\nu\left(C_T^1 + C_T^2\right) - \left(\nu\left(C_T^1\right) + \nu\left(C_T^2\right)\right)$$

$$= \frac{1}{2}\left(\nu\left(C_T^1/C_T^2\right) + \nu\left(C_T^2/C_T^1\right) - \nu\left(C_T^1\right) - \nu\left(C_T^2\right)\right)$$

Special cases

$C_T = C_T^1 + C_T^2$ with $C_T^1 = C^1(S_T, Y_T)$ and $C_T^2 = C^2(S_T)$

$$\boxed{\nu(C_T^1/C_T^2) = \nu(C_T^1) \quad \text{and} \quad \nu(C_T^2/C_T^1) = \nu(C_T^2)}$$

$C_T = C^1(S_T) + C^2(Y_T)$ with Y_T and S_T be independent under $\mathbb{P}$

$$\boxed{\nu(C_T^1/C_T^2) = \nu(C_T^1) \quad \text{and} \quad \nu(C_T^2/C_T^1) = \nu(C_T^2)}$$

Numeraire independence

Static no arbitrage constraint

$$\Updownarrow$$

Appropriate dependence across units needs to be built into the risk preference structure

$$\vdots$$

$$\Big\downarrow$$

Indifference pricing systems

$$\Updownarrow$$

Term structure of preferences

Numeraire independence (MZ 2004)

- Indifference prices in spot and forward units

- Indifference prices and state dependent risk tolerance

- Indifference prices and general numeraires

Indifference prices in spot and forward units

- Spot units

Wealth: $X_T^s = x + \alpha\left(\frac{S_T}{1+r} - S_0\right)$

Value function: $V^{s,C_T} = \sup_\alpha E_{\mathbb{P}}\left(-e^{-\gamma^s(X_T^s - \frac{C_T}{1+r})}\right)$

Pricing condition: $V^{s,0}(x) = V^{s,C_T}(x + \nu^s(C_T))$

Pricing measure: $E_{\mathbb{Q}^s}\left(\frac{S_T}{1+r}\right) = S_0$ and $\mathbb{Q}^s(Y_T|S_T) = \mathbb{P}^s(Y_T|S_T)$

Indifference price: $\nu^s(C_T) = \mathcal{E}_{\mathbb{Q}^s}\left(\frac{C_T}{1+r}\right) = E_{\mathbb{Q}^s}\left(\frac{1}{\gamma^s}\log E_{\mathbb{Q}^s}\left(e^{\gamma^s\frac{C_T}{1+r}}|S_T\right)\right)$

- Forward units

Wealth: $X_T^f = X_T^s(1+r) = f + \alpha(F_T - F_0)\ ; \quad f = x(1+r)$

Value function: $V^{f,C_T}(f) + \sup_\alpha E_{\mathbb{P}}\left(-e^{-\gamma^f(X_T^f - C_T)}\right)$

Pricing condition: $V^{f,0}(f) = V^{f,C_T}(f + \nu^f(C_T))$

Pricing measure: $E_{\mathbb{Q}^f}(F_T) = F_0$ and $\mathbb{Q}^f(Y_T|F_T) = \mathbb{P}(Y_T|F_T)$

Indifference price: $\nu^f(C_T) = \mathcal{E}_{\mathbb{Q}^f}(C_T) = E_{\mathbb{Q}^f}\left(\frac{1}{\gamma^f}\log E_{\mathbb{Q}^f}\left(e^{\gamma^f C_T}|F_T\right)\right)$

Consistency across spot and forward units

$$\nu^f(C_T) = (1+r)\nu^s(C_T) \Longleftrightarrow \delta^s = \frac{1}{1+r}\delta^f$$

$$\delta^s = \frac{1}{\gamma^s}\,, \quad \delta^f = \frac{1}{\gamma^f} \quad : \quad \text{spot and forward risk tolerance}$$

Risk tolerance is not a number. It is expressed in wealth units.

- Value function representations

$$V^{s,C_T}(x) = -e^{-\gamma^s(x-\nu^s(C_T))-H(\mathbb{Q}|\mathbb{P})} = U^s\left(x - \nu^s(C_T) + \frac{1}{\delta^s}H(\mathbb{Q}|\mathbb{P})\right)$$

$$V^{f,C_T}(x) = -e^{-\gamma^f(x-\nu^f(C_T))-H(\mathbb{Q}|\mathbb{P})} = U^f\left(x - \nu^f(C_T) + \frac{1}{\delta^f}H(\mathbb{Q}|\mathbb{P})\right)$$

$$\mathbb{Q} = \mathbb{Q}^s = \mathbb{Q}^f$$

Indifference prices and state dependent risk tolerance

- $\gamma_T = \gamma(S_T)$ or, equivalent, $\delta_T = \frac{1}{\gamma_T} = \delta(S_T)$

$$\boxed{\nu(C_T;\gamma_T) = \mathcal{E}_{\mathbb{Q}}\left(\frac{C_T}{1+r}\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma_T}\log E_{\mathbb{Q}}\left(e^{\gamma_T\frac{C_T}{1+r}}|S_T\right)\right)}$$

- Value functions representations

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}}(w) = \frac{\delta_T(w)}{E_{\mathbb{Q}}(\delta_T)}$$

$$V^{C_T}(x;\delta_T) = -\exp\left(-\frac{1}{E_{\mathbb{Q}}(\delta_T)}(x-\nu(C_T;\gamma_T)) - H(\mathbb{Q}^*|\mathbb{P})\right)$$

Consistency across value

The risk aversion process $\gamma_t,\, t = 0, T$ must satisfy

$$\boxed{E_{\mathbb{Q}^*}\left(\gamma_T \, \frac{S_T}{1+r}\right) = \gamma_0 S_0}$$

Q^* : has minimal entropy w.r.t. $\mathbb{P}$ but it is not a martingale measure

Optimal policies and risk monitoring

$$\boxed{\alpha^{*,C_T} = \alpha^{0,*} + \alpha^{1,*} + \alpha^{2,*}}$$

- $\alpha^{0,*} = -\dfrac{\partial H(\mathbb{Q}^*|\mathbb{P})}{\partial S_0} E_{\mathbb{Q}}(\delta_T)$ $\longleftarrow$ model incompleteness

- $\alpha^{1,*} = \dfrac{\frac{\partial}{\partial S_0} E_{\mathbb{Q}}(\delta_T)}{E_{\mathbb{Q}}(\delta_T)} x$ $\longleftarrow$ preference structure

- $\alpha^{2,*} = E_{\mathbb{Q}}(\delta_T) \dfrac{\partial}{\partial S_0} \left(\dfrac{\nu(C_T; \gamma_T)}{E_{\mathbb{Q}}(\delta_T)} \right)$ $\longleftarrow$ claim

Indifference prices and general numeraires

- The stock as the numeraire

Wealth: $$X_T^S = \frac{x(1+r)}{S_T} + \alpha \left(1 - \frac{S_0}{S_T}(1+r)\right)$$

Value function: $$V^{S,C_T} = \sup_{\alpha} E_{\mathbb{P}} \left(-e^{-\gamma^S (S_T)(X_T^S - \frac{C_T}{S_T})}\right)$$

Pricing condition: $$V^{S,0}(x^S) = V^{S,C_T}(x^S + \nu^S(C_T))$$

Pricing measure: $$\mathbb{Q}^S(Y_T|S_T) = \mathbb{P}(Y_T|S_T) \; ; \quad \frac{B_t}{S_t} \text{ martingale w.r.t. } \mathbb{Q}^S$$

Indifference price

$$\nu^S(C_T) = E_{\mathbb{Q}^S}\left(\frac{1}{\gamma^S(S_T)} \log E_{\mathbb{Q}^S}\left(e^{\gamma^S(S_T)\frac{C_T}{S_T}} \mid S_T\right)\right)$$

Numeraire consistency

$$\frac{\nu(C_T;\gamma_T)}{S_0} = \nu^S(C_T;\gamma_T^S) \quad \Longleftrightarrow \quad \delta_T = \delta_T^S \frac{S_T}{1+r}$$

Multiperiod case and the valuation algorithm

Need to construct a valuation scheme such that

$$\boxed{\nu_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{(t,T)}(C_T)}$$

- $\mathcal{E}_{\mathbb{Q}}^{(t,T)}(\cdot)$ should be a semigroup operator so that the above pricing scheme is time consistent

$$\boxed{\nu_t(C_T) = \nu_t(\nu_s(C_T)) \text{ for } t \leq s \leq T}$$

- Single pricing measure $\mathbb{Q}$.

- The pricing operator must be translation invariant w.r.t. hedgeable risks

$$\nu_t(c_1(S_T) + c_2(S_T, Y_T)) = E_{\mathbb{Q}}(c_1(S_T)) + \mathcal{E}_{\mathbb{Q}}^{(t,T)}(c_2(S_T, Y_T))$$

Given the inherent nonlinearity of the pricing mechanism is not at all obvious how the above pricing ingredients can be established!

The multiperiod model

– Traded asset: S_t, $t = 0, 1, ..., T$ $(S_t > 0, \forall t)$

$\xi_{t+1} = \dfrac{S_{t+1}}{S_t}$, $\xi_{t+1} = \xi^d_{t+1}, \xi^u_{t+1}$ with $0 < \xi^d_{t+1} < 1 < \xi^u_{t+1}$

– Second traded asset is riskless yielding zero interest rate

– Nontraded asset: Y_t, $t = 0, 1, ..., T$

$\eta_{t+1} = \dfrac{Y_{t+1}}{Y_t}$, $\eta_{t+1} = \eta^d_{t+1}, \eta^u_{t+1}$ with $\eta^d_t < \eta^u_t$

– $\{S_t, Y_t : t = 0, 1, ..., T\}$: a two-dimensional stochastic process

• Probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$

Filtrations $\mathcal{F}^S_t$ and $\mathcal{F}^Y_t$: generated by the random variables S_s (ξ_s) and Y_s (η_s), for $s = 0, 1, \ldots, t$.

Indifference pricing mechanism

- State wealth process: $X_s,\ s = t+1, ..., T$

  $\alpha_s,\ s = t+1, t+2, ..., T$: the number of shares of the traded asset held in this portfolio over the time period $[s-1, s]$

$$X_T = x + \sum_{s=t+1}^{T} \alpha_s \triangle S_s$$

- Claim C_T (Path dependence/early exercise are allowed)
- Value function: $V^{C_T}(X_t, t; T) = \sup_{\alpha_{t+1},...,\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T)} \mid \mathcal{F}_t\right)$
- Indifference price: $\nu_t(C_T)$

$$\boxed{V^0(X_t, t; T) = V^{C_T}(X_t + \nu_t(C_T), t; T)}$$

Fundamental multiperiod pricing blocks

Let Z_s, $0 \leq s \leq t$ be an $\mathcal{F}_s$-adapted process and Q a martingale measure

$$\boxed{\mathcal{E}_Q^{(t,s)}(Z_s) \triangleq \mathcal{E}_Q^{(t,s-1)}(\mathcal{E}_Q^{(s-1,s)}(Z_s))}$$

where

$$\boxed{\mathcal{E}_Q^{(s-1,s)}(Z_s) \triangleq E_Q\left(\tfrac{1}{\gamma}\log E_Q(e^{\gamma Z_s}|\mathcal{F}_{s-1} \vee \mathcal{F}_s^S)|\mathcal{F}_{s-1}\right)}$$

and

$$\boxed{\mathcal{E}_Q^{(s,s)}(Z_s) = Z_s}$$

- Note that for $t < s-1$

$$\mathcal{E}_Q^{(t,s)}(Z_s) \neq E_Q\left(\frac{1}{\gamma}\log E_Q(Z_s|\mathcal{F}_t \vee \mathcal{F}_s^S)|\mathcal{F}_t\right)$$

The valuation algorithm

Let $\mathbb{Q}$ be a martingale measure satisfying, for $t = 0, 1, \ldots, T$

$$\mathbb{Q}(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1}) = \mathbb{P}(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1})$$

- The indifference price $\nu_t(C_T)$ satisfies

$$\nu_t(C_T) = \mathcal{E}^{(t,t+1)}_{\mathbb{Q}}(\nu_{t+1}(C_T)) \qquad \nu_T(C_T) = C_T$$

- The indifference price process is given by

$$\nu_t(C_T) = \mathcal{E}^{(t,T)}_{\mathbb{Q}}(C_T)$$

- The pricing algorithm is consistent across time

$$\nu_t(C_T) = \mathcal{E}^{(t,s)}_{\mathbb{Q}}(\mathcal{E}^{(s,T)}_{\mathbb{Q}}(C_T)) = \mathcal{E}^{(t,s)}_{\mathbb{Q}}((\nu_s(C_T)) = \nu_t(\mathcal{E}^{(s,T)}_{\mathbb{Q}}(C_T))$$

Properties of the iterative pricing functional $\mathcal{E}_Q^{(t,T)}(\cdot)$

For $0 \leq t \leq s \leq T$

i) $\mathcal{E}_Q^{(t,s)}(Z_s)$ is an $\mathcal{F}_t$-adapted process

ii) $\mathcal{E}_Q^{(t,T)}(Z_s) = \mathcal{E}_Q^{(t,s)}(Z_s)$

iii) $\mathcal{E}_Q^{(t,s)}(Z_s) = \mathcal{E}_Q^{(t,t+1)}(\ldots \mathcal{E}_Q^{(s-2,s-1)}(\mathcal{E}_Q^{(s-1,s)}(Z_s)))$

iv) $\mathcal{E}_Q^{(t,s)}(Z_s) = \mathcal{E}_Q^{(t,s-t)}(\mathcal{E}_Q^{(s-t,s)}(Z_s))$

Interpretation of the pricing algorithm

Valuation is done via an iterative pricing scheme applied backwards in time.

$$\nu_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{(t,t+1)}(\mathcal{E}_{\mathbb{Q}}^{(t+1,t+2)}(\ldots(\mathcal{E}_{\mathbb{Q}}^{(T-1,T)}(C_T)))$$

$$\nu_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{(t,t+1)}(\nu_{t+1}(C_T))$$

– **Specification, isolation and pricing of unhedgeable risks**

$\nu_{t+1}(C_T)$: the end of the period payoff is altered to the preference adjusted payoff

$$\tilde{\nu}_{t+1}(C_T) = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma \nu_{t+1}(C_T)} \middle| \mathcal{F}_t \vee \mathcal{F}_{t+1}^S \right)$$

– **Pricing of remaining hedgeable risks by arbitrage**

$$\nu_t(C_T) = E_{\mathbb{Q}}(\tilde{\nu}_{t+1}(C_T))$$

Conclusions

- The single step valuation functional $\mathcal{E}_{\mathbb{Q}}^{(t,t+1)}(.)$

  Nonlinear

  Payoff independent

  Preference dependent in the first step only

- The pricing measure $\mathbb{Q}$

$$\mathbb{Q}\left(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right) = \mathbb{P}\left(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right), \quad t = 0, ...T-1$$

  Preserves the conditional distributions of unhedgeable risks, given the hedgeable ones, from their historical values

Continuous time models

The Markovian case

- Probabilistic setting
  - Traded asset: $S_t, \quad dS_s = \mu(S_s, s)S_s ds + \sigma(S_s, s)S_s dW_s^1$
  - Riskless bond B_t
  - Nontraded asset: $Y_t, \quad dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s$
  - Probability space: $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$

$$\mathcal{F}_s = \sigma(W_u^1, W_u; 0 \leq u \leq s) \qquad\qquad d(W^1, W)(s) = \rho ds$$

  - Claim: $C_T = C(S_T, Y_T)$

Indifference price representation

The indifference price process is given by

$$\boxed{\nu_s(C_T) = H(S_s, Y_s, s)}$$

where $H(S, y, t)$ is the unique viscosity solution of the quasilinear price equation

$$H_t + \mathcal{L}(H(S, y, t)) + \frac{1}{2}\gamma(1 - \rho^2)a^2(y, t)H_y^2 = 0$$

$$H(S, y, T) = C(S, y)$$

The operator $\mathcal{L}(.)$ is given by

$$\mathcal{L}(.) \;=\; \frac{1}{2}\sigma^2(S, t)\frac{\vartheta^2}{\vartheta S^2} + \rho a(y, t)\sigma(S, t)\frac{\vartheta^2}{\vartheta S \vartheta y} + \frac{1}{2}a^2(S, t)\frac{\vartheta^2}{\vartheta y^2}$$

$$+ \left(b(y, t) - \rho\frac{\mu(S, t)}{\sigma(S, t)}a(y, t)\right)\frac{\vartheta}{\vartheta y}$$

- Observation 1

  i) H may be directly related to a quadratic stochastic control problem

  ii) H may be related to a BSDE

  But, there is **no** direct Feynman-Kac type representation of H available up to now!

- Observation 2

  Solutions of the above equation are directly structurally related to optimal investment policies in models with stochastic Sharpe ratio, recursive utilities, and stochastic labor income

Stoikov and Zariphopoulou (2004)

Special case

(Musiela and Zariphopoulou (2004))

- Lognormal dynamics for the traded asset S_t :

$$S_s = \mu S_s ds + \sigma S_s dW_s^1$$

- Payoff depending exclusively on the nontraded asset Y_t

$$C(S_T, Y_T) = C(Y_T)$$

$$H(y,t) = \frac{1}{\gamma(1-\rho^2)} \ln\left(E_{\mathbb{Q}}(e^{\gamma(1-\rho^2)C(Y_T)} \mid Y_t = y) \right)$$

The above formula is **not** a naive extension of the classical certainty equivalent price!

Valuation algorithm

- **Pricing measure $\mathbb{Q}$:** a martingale measure satisfying

$$\mathbb{Q}(Y_{t+dt} \Big| \mathcal{F}_t \vee \mathcal{F}^S_{t,t+dt}) = \mathbb{P}(Y_{t+dt} \Big| \mathcal{F}_t \vee \mathcal{F}^S_{t,t+dt})$$

The indifference price $\nu_t(S_s, Y_s, s)$ is given by the iterative algorithm

$$\nu_t(S_s, Y_s, s) =$$

$$= E_{\mathbb{Q}}\left(\frac{1}{\gamma} \log E_{\mathbb{Q}}(e^{\gamma\nu(S_{s+ds}, Y_{s+ds}, s+ds)} \mid \mathcal{F}_s \vee \mathcal{F}^S_{s,s+ds}) \mid \mathcal{F}_s\right)$$

$$\nu_T(S_T, Y_T, T) = C(S_T, Y_T)$$

A novel probabilistic representation of solutions to quasilinear pdes arises

The solution $H(S,y,t)$ of the quasilinear price equation

$$H_t + \mathcal{L}(H(S,y,t)) + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)H_y^2 = 0$$

with

$$\mathcal{L}(.) = \frac{1}{2}\sigma^2(S,t)\frac{\vartheta^2}{\vartheta S^2} + \rho a(y,t)\sigma(S,t)\frac{\vartheta^2}{\vartheta S \vartheta y} + \frac{1}{2}a^2(S,t)\frac{\vartheta^2}{\vartheta y^2}$$

$$+ \left(b(y,t) - \rho\frac{\mu(S,t)}{\sigma(S,t)}a(y,t)\right)\frac{\vartheta}{\vartheta y}$$

is given by

$$\boxed{H(S,y,t) = \lim_{ds\to 0, s\to t} \nu_t(S_{s+ds}, Y_{s+ds}, s+ds)}$$

Proof: Based on convergence results of supermartingales, behavior of nonadditive measures and robustness properties of viscosity solutions.

Extensions

- Early exercise claims: $C_\mathcal{T} = C(S_\mathcal{T}, Y_\mathcal{T})$; $\mathcal{T} \in \mathcal{F}_{[0,T]}$

$$\nu_t^a(C_\mathcal{T}) = \sup_{\mathcal{T} \in \mathcal{F}_{[0,T]}} \nu_t(C_\mathcal{T}) = \sup_{\mathcal{T} \in \mathcal{F}_{[0,T]}} (\mathcal{E}_{\mathbb{Q}}^{(t,\mathcal{T})}(C_\mathcal{T}))$$

  (Musiela and Zariphopoulou (2003) and Oberman and Zariphopoulou (2003))

- Path dependent claims: $C_{(t,T)} = C(\int_t^T c(S_s, Y_s, s)ds; \omega)$

- Stochastic interest rates (infinite dimensional problems/PSDEs)

  (Tehranchi and Zariphopoulou)

- Defaultable claims (Bielecki et. al (2004))

Sketch of the proof:

- **Step 1:** Establish

$$\nu_{T-1}\left(C_T\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\left|\mathcal{F}_{T-1}\vee\mathcal{F}_T^S\right.\right)|\mathcal{F}_{T-1}\right)$$

Observe

$$\begin{aligned}V^{C_T}(X_{T-1},T-1;T) &= \sup_{\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_T-C_T)}\,|\mathcal{F}_{T-1}\right)\\ &= \sup_{\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_{T-1}+\alpha_T\triangle S_T-C_T)}\,|\mathcal{F}_{T-1}\right)\end{aligned}$$

and deduce that

$$V^{C_T}(X_{T-1},T-1;T) = -e^{-\gamma X_{T-1}-h_{T-1}+\gamma\lambda_{T-1}(C_T)}$$

where

$$h_{T-1} = q_T\log\frac{q_T}{\mathbb{P}(A_T\mid\mathcal{F}_{T-1})} + (1-q_T)\log\frac{1-q_T}{1-\mathbb{P}(A_T\mid\mathcal{F}_{T-1})}\ ,$$

$$A_T = \{\omega : \xi_T(\omega) = \xi_T^u\}$$

and

$$\lambda_{T-1}(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\middle|\mathcal{F}_{T-1} \vee \mathcal{F}_T^S\right)|\mathcal{F}_{T-1}\right)$$

Similarly,

$$V^0(X_{T-1}, T-1; T) = -e^{-\gamma X_{T-1} - h_{T-1}}$$

which yields that

$$\nu_{T-1}(C_T) = \lambda_{T-1}(C_T)$$

$$= E_{\mathbb{Q}}\left(\tfrac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\middle|\mathcal{F}_{T-1} \vee \mathcal{F}_T^S\right)|\mathcal{F}_{T-1}\right) = \mathcal{E}_{\mathbb{Q}}^{(T-1,T)}(C_T)$$

- **Step 2:** Establish the semigroup property for two time steps

$$\nu_{T-2}(C_T) = \mathcal{E}_{\mathbb{Q}}^{(T-2,T-1)}(\nu_{T-1}(C_T)),$$

with $\nu_{T-1}\left(C_T\right)$ given by

$$\nu_{T-1}\left(C_T\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\left|\mathcal{F}_{T-1} \vee \mathcal{F}_T^S\right)\right|\mathcal{F}_{T-1}\right)$$

Construct $\nu_{T-2}(C_T)$ directly via the fundamental pricing formula

$$V^{C_T}(X_{T-2}, T-2; T) = \sup_{\alpha_{T-1},\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T)}\left|\mathcal{F}_{T-2}\right.\right)$$

$$= \sup_{\alpha_{T-1}} E_{\mathbb{P}}\left(\sup_{\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_{T-1}+\alpha_T \triangle S_T - C_T)}\left|\mathcal{F}_{T-1}\right.\right)\left|\mathcal{F}_{T-2}\right.\right)$$

$$= -e^{-\gamma X_{T-2} - E_{\mathbb{Q}}(h_{T-2}+h_{T-1}|\mathcal{F}_{T-2}) + \gamma\lambda_{T-2}(C_T t)},$$

with

$$\lambda_{T-2}(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\nu_{T-1}(C_T)} \mid \mathcal{F}_{T-2} \vee \mathcal{F}^S_{T-1}\right) \mid \mathcal{F}_{T-2}\right)$$

Observe that

$$\lambda_{T-2}(C_T) = \mathcal{E}^{(T-2,T-1)}(\nu_{T-1}(C_T))$$

Similarly, we deduce that

$$V^0(X_{T-2}, T-2; T) = -e^{-\gamma X_{T-2} - E_{\mathbb{Q}}(h_{T-2}+h_{T-1}|\mathcal{F}_{T-2})}$$

Conclude that

$$\boxed{\nu_{T-2}(C_T) = \lambda_{T-2}\left(C_T\right) = \mathcal{E}^{(T-2,T-1)}_{\mathbb{Q}}(\nu_{T-1}(C_T))}$$

Construction of the single period price

$$\boxed{V^0(x) = V^{C_T}(x + \nu(C_T))}$$

$$V^{C_T}(x) = \max_\alpha E_{\mathbb{P}}(-e^{-\gamma(X_T - C_T)}) \ , \quad V^0(x) = \max_\alpha E_{\mathbb{P}}(-e^{-\gamma X_T})$$

Model ingredients

– Historical probabilities $p_i, \ \ i = 1, .., 4$

$$p_1 = \mathbb{P}(S^u, Y^u) \quad p_2 = \mathbb{P}(S^u, Y^d) \quad p_3 = \mathbb{P}(S^d, Y^u) \quad p_4 = \mathbb{P}(S^d, Y^d)$$

– Nested risk-neutral probabilities q_i: $\quad q_1 + q_2 = \dfrac{S_0 - S^d}{S^u - S^d}$

– Payoff values $\quad C_T(\omega) = C(S_T(\omega), Y_T(\omega)) \longleftrightarrow c_i, \quad i = 1, ..., 4$

– Risk aversion

Standard optimization arguments yield

$$V^{C_T}(x) = -\frac{e^{-\gamma x}}{q^q(1-q)^{1-q}}(e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2)^q(e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4)^{1-q}$$

$$V^0(x) = -\frac{e^{-\gamma x}}{q^q(1-q)^{1-q}}(p_1 + p_2)^q(p_3 + p_4)^{1-q}$$

Indifference price formula

$$\boxed{\nu(C_T) = \frac{1}{\gamma}\left(q\log\frac{e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2}{p_1 + p_2} + (1-q)\log\frac{e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4}{p_3 + p_4}\right)}$$

Probabilistic interpretation of the indifference price

$$\nu(C_T) = q\left(\frac{1}{\gamma}\log\frac{e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2}{p_1+p_2}\right) + (1-q)\left(\frac{1}{\gamma}\log\frac{e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4}{p_3+p_4}\right)$$

Key observation

$$\frac{e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2}{p_1+p_2} \quad \leftarrow \quad E_{\mathbb{P}}(e^{\gamma C(S_T,Y_T)} \mid S_T = S^u)$$

$$\frac{e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4}{p_3+p_4} \quad \leftarrow \quad E_{\mathbb{P}}(e^{\gamma C(S_T,Y_T)} \mid S_T = S^d)$$

$$\nu(C_T) = E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\log(E_{\mathbb{P}}(e^{\gamma C(S_T,Y_T)} \mid S_T))\right)$$

Choice of pricing measure $\mathbb{Q}$

- $\mathbb{Q}$ needs to be a **martingale measure**
- $\mathbb{Q}$ needs to **preserve the conditional distribution of the unhedgeable risks, given the hedgeable ones, from their historical values**

$$\boxed{\mathbb{Q}(Y_T \mid S_T) = \mathbb{P}(Y_T \mid S_T)}$$

Indifference price components

$$\tilde{C}_T = \frac{1}{\gamma} \log \left(E_{\mathbb{P}}(e^{\gamma C(S_T, Y_T)} | S_T) \right)$$

$$= \frac{1}{\gamma} \log \left(E_{\mathbb{Q}}(e^{\gamma C(S_T, Y_T)} | S_T) \right)$$

$$\nu(C_T) = E_{\mathbb{Q}}(\tilde{C}_T)$$

Choice of the pricing measure

Let $\mathcal{Q}_e$ be the set of martingale measures

For any $Q \in \mathcal{Q}_e$, the joint distributions of $(\xi_1, \xi_2, ...\xi_T, \eta_1, \eta_2, ...\eta_T)$ can be computed through the conditionals

$$Q(\xi_1, ...\xi_T, \eta_1, ...\eta_T) = \prod_{s=0}^{T-1} Q(\xi_{s+1}, \eta_{s+1} \,|\mathcal{F}_s)$$

However, $$Q(\xi_{t+1}, \eta_{t+1} \,|\mathcal{F}_t) = Q(\xi_{t+1} \,|\mathcal{F}_t) Q(\eta_{t+1} \Big| \mathcal{F}_t \vee \mathcal{F}^S_{t+1})$$

with the term $Q(\xi_{t+1} \,|\mathcal{F}_t)$ depending exclusively on ξ^u_{t+1} and ξ^u_{t+1}

Therefore, the choice of pricing measure amounts only to the specification of the last term,

$$Q(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1})$$

This conditional distribution essentially reflects the "probabilistic vision" of the unhedged risks that the pricing mechanism must carry through

But, given their nature, these **risks, need to be "viewed" as identical both by the real and the candidate pricing measure**

$$\boxed{\mathbb{Q}(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1}) = \mathbb{P}(\eta_{+1} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1})}$$

Interesting observation

A martingale measure $\mathbb{Q} \in Q_e$ satisfies

$$\mathbb{Q}\left(\eta_{t+1} \middle| \mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right) = \mathbb{P}\left(\eta_{t+1} \middle| \mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right)$$

if and only if it has the minimal, relative to $\mathbb{P}$, entropy

- Characterization of minimal entropy measure

- Intuitive representation of the indifference pricing measure