THE OBSTACLE PROBLEM REVISITED

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The obstacle problem consists in studying the properties of minimizers of the Dirichlet integral

$$D(u) = \int_D (\nabla u)^2 \, dX$$

in a domain, D, of R^n, among all those configurations $u(X)$, with prescribed boundary values: $u|_{\partial D} = f(X)$, and constrained to remain, in D, above a prescribed obstacle $\varphi(X)$.

More precisely, we are given:

a) A (smooth) domain, D, of R^n.

b) A (smooth) function $f(X)$ on ∂D.

c) A (smooth) function $\varphi(X)$ on D, with $\varphi|_{\partial D} < f(X)$.

In the Hilbert space, $H^1(D)$, of all those functions, u, with square integrable gradient, we define K to be the closed convex set

$$K = \{u \in H^1,\ u|_{\partial D} = f(X),\ u \geq \varphi\}$$

On K, there is a unique point u_0 that minimizes the Dirichlet integral

$$D(u) = \int (\nabla u)^2 \, dX \ .$$

Such a point u_0 is called the "solution to the obstacle problem." Such a problem is motivated by the description of the equilibrium position of a membrane (the graph of u) that is "attached" at level $f(X)$ along the boundary of D, and is restricted to remain above φ, (the obstacle).

Such a membrane will minimize area integral

$$A(u) = \int \sqrt{1 + (\nabla u)^2}\, dX$$

that is linearized to Dirichlet integral for small deflections. In any case, the theory developed here applies to the "minimal surface", i.e., the non linearized case, but for simplicity we will restrict here to the "linear" case.

The same mathematical problem appears in many other contexts: fluid filtration in porous media, elasto-plasticity, optimal control and financial math. See for instance the book of Friedman ([Fr]) where many of these applications are described, as well as the classical literature on this problem.

In this article we will review the classical regularity theory of solution and free boundary, [F], [C1], [C2], [C3], and present some new facts and proofs, in particular the description of the structure of the singular set.

I developed most of this theory during my first years at Minnesota, and I fondly remember Gene's encouragement and help.

We start with some classical statements about u_0:

Lemma 1.

a) *u_0 stays between $\lambda_1 = \min f(X)$, and $\lambda_2 = \max(f(X), \varphi(X))$.*

b) *u_0 is superharmonic, and support $\Delta u_0 \subset \{u_0 = \varphi\}$.*

Point a) is a standard application of the weak maximum principle for H^1 functions.

The minimum of two such functions is again in H^1, so, for instance, $\bar{u} = \min(u_0, \lambda_2(= \max f(X), \varphi(X))$, is an admissible function and

$$D(u_0) = D(\bar{u}) + \int_{\{u_0 > \lambda_2\}} (\nabla u_0)^2\, dX \ .$$

Therefore $u_0 = \bar{u}$, a.e.

About b) u_0 is subharmonic. Indeed any positive $\varphi \in C_0^1$ is an admissible perturbation, i.e.,

$$\int (\nabla u_0)^2 \, dX \le \int (\nabla u_0 + \varepsilon\varphi)^2$$
$$= \int (\nabla u_0)^2 + 2\varepsilon \int \nabla u_0 \nabla \varphi + \varepsilon^2 \int (\nabla \varphi)^2 \ .$$

Therefore

$$0 \le \int \nabla u_0 \nabla \varphi \ .$$

Corollary 1. *u_0 is pointwise defined, semicontinuous, and the set $\{u_0 > \varphi\}$ is open. (More precisely if $u_0(X_0) \ge \varphi(X_0) + \delta$, there exists a neighborhood of X_0, where $u_0(X) \ge \varphi(X) + \delta/2$.)*

Proof. It follows from the mean value property.

Corollary 2. *Δu_0 is supported in the closed set $\{u_0 = \varphi\}$.*

Corollary 3. *u_0 is continuous.*

The proof of Corollary 3 is a consequence of Evans theorem, that says:

Theorem 1. *Let v be a superharmonic function, and suppose that $v|\mathrm{support}(\Delta v)$ is continuous. Then v is continuous.*

Let me sketch the *proof of the Theorem 1*. Suppose that there exists a sequence $X_k \to X_0$, such that $\lim v(X_k) \ne v(X_0)$.

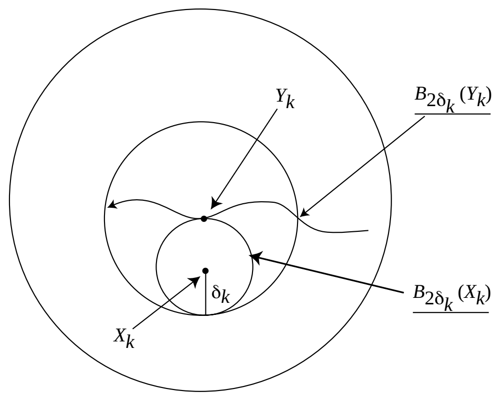
Then a) $X_0 \in \mathrm{support}(\Delta v)$, if not v would be harmonic in a neighborhood of X_0 and thus continuous. (Remember that by definition, $\mathrm{support}(\Delta v)$ is closed.)

b) $X_k \notin \mathrm{support}(\Delta v)$, since v is continuous there.

c) $\lim v(X_k) = a > v(X_0)$ (no loss of generality) $= 0$ by semicontinuity.

d) By semicontinuity, given $\varepsilon > 0$, we may assume $v(X) \ge -\varepsilon$, for $|X - X_0| \le \delta$.

Let Y_k be the closest point to X_k in support(Δv) (in particular $\delta_k = |Y_k - Y_k| \le |X_0 - X_k| \to 0$.)



Then $v(Y_k) \to v(X_0) = 0$. Let us find a contradiction: By superharmonicity

$$v(Y_k) \ge \frac{1}{|B_{2\delta_k}(Y_k)|} \int_{B_{2\delta_k}(Y_k)} v(Y)\, dY \ .$$

But we evaluate

$$\int_{B_{2\delta_k}(Y_k)} v(y)\, dY = \int_{B_{2\delta_k}(Y_k) \setminus B_{\delta_k}(X_k)} + \int_{B_{\delta_k}(X_k)} = I_1 + I_2 \ .$$

For I_1, we use that $v \ge -\varepsilon$, once our configuration is close to X_0. Therefore $I_1 \ge -\varepsilon |B_{2\delta_k}(Y_k)|$.

For I_2, we use that v is harmonic in $B_{\delta_k}(X_k)$. Thus the mean value theorem holds and

$$I_2 = |B_{\delta_k}| v(X_k) \ .$$

Therefore

$$v(Y_k) \ge -\varepsilon + \frac{1}{2^n} v(X_k) \ .$$

that converges to $\frac{a}{2^n}$, a contradiction.

Before we go on, let me list a few *properties of harmonic and superharmonic functions* that are an easy consequence of the mean value theorem:

a) Harnack inequality.

If v is harmonic and non-negative in B_1, then for $R < 1$

$$\sup_{B_R} v \leq C(R) \inf_{B_R} v \ .$$

($C(R)$ goes to infinite when R goes to one as a (negative) power of $(1 - R)$).

b) Derivative estimates.

If v is harmonic in $B_1(0)$

$$|\nabla v(0)| \leq \operatorname*{osc}_{B_1(0)} v \ .$$

Note that harmonicity is a linear, translation invariant property, thus derivatives of harmonic functions are again harmonic, and by iterating b), and scaling, we get

b′) $|D^{(k)} v(0)| \leq C(k) \frac{1}{r^k} \operatorname*{osc}_{B_r(0)} v$.

c) Given a superharmonic function v in D, v cannot have a local minimum, $v(X_0)$ in D, unless v is identically a constant.

Finally, we will also need the solvability of Laplace's equation in a ball.

d) Given f continuous in ∂B_1, then there exists a unique harmonic function, v, such that

$$v|_{\partial B_1} = f \ .$$

We are now ready to launch into the study of the regularity properties of u and the set $u = \varphi$. We start with the regularity of u. The $C^{1,1}$ regularity of u, when φ is $C^{1,1}$ is due to Frehse [F]. The proof below is in [C-K]. We will always stay away from ∂D.

Theorem 2. *"Up to $C^{1,1}$, u is as regular as φ". More precisely*

a) *Assume that φ has a modulus of continuity $\sigma(r)$, then u has modulus of continuity $C\sigma(2r)$*

b) *Assume now that $\nabla\varphi$ has modulus of continuity $\sigma(r)$, then ∇v has modulus $C\sigma(2r)$.*

To prove Theorem 2, we start with

Lemma 2. *Let $u(X_0) = \varphi(X_0)$. Then "u separates from φ with the speed dictated by $\sigma(r)$", more precisely in case a)*

$$\sup_{B_r(X_0)} u - \varphi \leq C\sigma(2r) \ .$$

In case b)

$$\sup_{B_r(X_0)} u - \varphi \leq Cr\sigma(2r) \ .$$

Proof. We prove case b). Let $L = \varphi(X_0) + \langle \nabla\varphi(X_0), X - X_0 \rangle$ be the linear part of φ at X_0. Then, by definition of $\sigma(r)$, on $B_r(X_0)$

$$L - r\sigma(r) \leq \varphi(X) \leq u(X) \ .$$

Let us show that on $B_{r/2}$,

$$u(X) \leq L + Cr\sigma(r) \ .$$

Consider

$$w = u - (L - r\sigma(r)) \ .$$

This is a non-negative superharmonic function in B_r.

Let us split it in $w = w_1 + w_2$, with w_1 harmonic and equal to w in ∂B_r.

Thus

$$0 \leq w_1 \leq w \ ,$$

and hence also

$$0 \leq w_2 \leq w \ .$$

We have that

$$w_1(X_0) \leq u(X_0) - (L - r\sigma(r)) = \varphi(X_0) - (L - r\sigma(r)) = r\sigma(r) \ .$$

By Harnack inequality

$$w_1|_{B_{r/2}} \leq Cr\sigma(r) \ .$$

About w_2, it is superharmonic and vanishes on ∂B_r. Thus, it attains its maximum in the support of its Laplacian. But

$$\Delta w_2 = \Delta u \ .$$

So it attains its maximum at a point X_1, where $u = \varphi$.

But $w_2 \le w = u - (L - r\sigma(r))$. Thus

$$w(X_1) \le \varphi(X_1) - (L - r(\sigma(r)) \le C\sigma(r) \ .$$

The proof of the lemma is complete.

In particular if for instance φ is $C^{1,1}$, u "lifts away" from φ in a quadratic fashion, that is $(u - \varphi)(X) \le C|X - X_0|^2$.

From part b) of the Theorem, let me just show that if φ is $C^{1,1}$, then u is $C^{1,1}$ (away from ∂D). This just follows by scaling: Let $X_1 \in \Omega = \{u > \varphi\}$, $d(X_1, \Lambda) = d(X, X_0) = \rho$ ($\Lambda = \{u = \varphi\}, X_0 \in \Lambda$). Then on $B_\rho(X_1)$, u is harmonic. But on $B_{4\rho}(X_1)$, u has quadratic bounds away from L_{X_0}. (Since, φ being $C^{1,1}$, we can take $r\sigma(r) = Cr^2$.) Thus,

$$\|D^2 u(X_1)\| \le \frac{1}{\rho^2} \operatorname*{osc}_{B_\rho(X_1)} (u - L_{X_0}) \le C \frac{1}{r^2} r^2 \le C \ .$$

We have now completed the local regularity theory of u, i.e., $u \in C^{1,1}$ is as good an estimate as we may hope for, since Δu jumps from zero to $\Delta\varphi$ across $\partial\Omega$.

We now begin to study the regularity of $\partial\Omega$, that is

Free Boundary Regularity: Part I, generalities.

The material in this part can be found in [C1] and [C3].

To study the free boundary regularity, we reduce it to a local problem, and consider the new variable $w = u - \varphi$. Then, we have the following local problem.

Definition (Normalized solutions). In the unit ball of R^n we are given a function w with the following properties:

a) $w \ge 0$, w is $C^{1,1}$.

b) On the set $\Omega = \{w > 0\}$, $\Delta w \equiv 1$

c) The point 0 belongs to $\partial\Omega$ (i.e., is a free boundary point).

Question. What can we say about the geometry of $\partial\Omega$?

Some Remarks. a). On $\Omega = \{w > 0\}$ we really have

$$\Delta w = \Delta(u - \varphi) = -\Delta\varphi = g(X) \ .$$

Since u is superharmonic, it cannot touch φ at a point where $\Delta\varphi \geq 0$, so near the free boundary we should expect $g(X) \geq 0$. In fact, if $\Delta\varphi$ and $\nabla\Delta\varphi$ do not vanish simultaneously (a necessary non-degeneracy condition), a variation of Hopf's principle shows that $g(X) > 0$ near the free boundary.

Remark b). We have made $g(X) \equiv 1$. All it is necessary is $g(X) =$ of class C^α for the general theory and $g(X)$ of class $C^{1,\alpha}$, to show that singular points lay in smooth manifolds, but these assumptions would fill the proofs with little technicalities.

Remark c): Important Rescaling Observations. The function $w_\lambda = \frac{1}{\lambda^2} w(\lambda X)$ satisfies the same conditions, in B_λ instead of B_1.

We start with the following

Lemma 3: Optimal regularity. *w restricted to $B_{1/2}$ is bounded by a universal constant and its $C^{1,1}$ norm is also bounded by a universal constant.*

Proof. Apply Theorem 2 to $u = w - \frac{1}{2n}|X|^2$, $\varphi = -\frac{1}{2n}|X|^2$.

Lemma 4: Optimal gradient bound.

$$|\nabla w(X_0)| \leq C(w(X_0))^{1/2} \ .$$

Proof. Let $w(X_0) = h > 0$. Then, from the second derivative bounds, $B_{(Ch)^{1/2}}(X_0) \subset \Omega$. (If $\exists\ X_1 \in B_{Ch^{1/2}}(X_0) \cap \Lambda$, find a contradiction!). In $B_{(Ch)^{1/2}}(X_0)$, $\Delta w = 1$ and

$$v = w + \frac{(Ch - |X - X_0|^2)}{2n}$$

is harmonic and non-negative.

From Harnack and Interior estimates

$$|\nabla w(X_0)| = |\nabla v(X_0)| \le \frac{C}{(Ch)^{1/2}} \operatorname*{osc}_{B_{(Ch)^{1/2}}} v \le \frac{C}{(Ch)^{1/2}} v(X_0)$$
$$\le \frac{C}{h^{1/2}} h = C h^{1/2} \ .$$

Lemma 5: (Maximum growth). *Let* $X_0 \in \overline{\Omega}$*, then* $\sup_{B_r(X_0)} w \ge Cr^2$*.*

Proof. It is enough, by continuity, to prove it for $X_0 \in \Omega$. Let $v = w - \frac{1}{2n}|X - X_0|^2$. Then v is harmonic in $\Omega \cap B_r(X_0)$, and positive at X_0. It should take a positive maximum on $\partial(\Omega \cap B_r(X_0))$. But on $(\partial\Omega) \cap B_r(X_0)$, $w \equiv 0$, so v is negative.

Thus the maximum takes place at a point $X_1 \in \partial B_r$. There

$$0 < v = w(X_1) - \frac{1}{2n} r^2 \ .$$

Lemma 6. *The "level strip",* $S_h = \{0 < w < h^2\} \subset \Omega$*, satisfies*

$$\operatorname{meas}(S_h \cap B_r) = |S_h \cap B_r| \le C h\, r^{n-1} \ .$$

We split the proof into two steps:

Lemma 6a. *Let* $w_e = D_e w$ *be the directional derivative of* w *in the direction* e*. Then*

$$\int_{\{0 \le w_e \le h\} \cap B_r} |\nabla w_e|^2 \le C h\, r^{n-1} \ .$$

Proof. By the rescaling properties of w (i.e., by looking at $w_r = \frac{1}{r^2} w(rX)$) it is enough to look at $r = 1$ (Check this fact!!). We truncate w_e at levels ε and h: $\overline{w}_e = \min[(w_e - \varepsilon)^+, h]$, and write the usual formula

$$\int_{B_1} \nabla \overline{w}_e \nabla w_e + \overline{w}_e \Delta w_e = \int_{\partial B_1} \overline{w}_\varepsilon D_\nu w_e \ .$$

Since $\Delta w_e \equiv 0$ in Ω (in particular for $w_e > \varepsilon$) and $|D_\nu w_e| \le |D^2 w| \le C$, we immediately get

$$\int_{\{\varepsilon < w_e < h\}} |\nabla w_e|^2 \le Ch \ .$$

Proof of Lemma 6. Again we prove it for $r = 1$

$$S_h \subset \{|\nabla w| < h\} \text{ (from Lemma 4) } \subset \cap \{w_{\pm e_n} < h\} \ .$$

Thus

$$|S_h \cap B_1| = \int_{S_h \cap B_1} \Delta w \le C \int_{S_h \cap B_1} |D^2 w|^2 \le \Sigma \int_{\{w_{\pm e_n} < h\}} |\nabla w_e|^2 \, dX \ .$$

Corollary 4. *The neighborhood N_δ of the free boundary*

$$(N_\delta(S)) = \{X : d(X, S) \le \delta\}$$

has measure

$$|N_\delta \cap B_r| \le \delta \, r^{n-1} \ .$$

In particular, the free boundary has locally finite $n-1$ directional Hausdorff measure, and

$$H^{n-1}(\partial\Omega \cap B_r) \le C \, r^{n-1} \ .$$

Proof. $(N_\delta \cap \Omega) \subset S_\delta$, and thus

$$|N_\delta \cap \Omega \cap B_r| \le \delta \, m r^{n-1} \ .$$

But Ω has uniform positive density along the free boundary, i.e., for $X_0 \in \partial\Omega$, we have

$$\frac{|B_r(X_0) \cap \Omega|}{|B_r|} \ge \mu > 0$$

for some universal constant μ.

Indeed, from Lemma 5 (maximum growth), $w(X_1) = \sup_{B_r(X_0)} w \geq Cr^2$, and from $C^{1,1}$ estimates, $|\nabla w|_{B_{2r(X_0)}} \leq Cr$. Thus w must remain positive in $B_{Cr}(X_1)$ for C small enough.

This completes the "generalities" part of our discussion, that is those properties and techniques common to many free boundary problems: optimal regularity, to allow for rescalings, maximum possible growth to provide stability of the free boundaries and coincidence sets under rescalings and measure theoretical properties of the free boundaries.

Free Boundary Regularity: Part II.

We now go to a special issue in each free boundary problem, that is the classification of global solutions to our problem. I.e., if we plan, as in the theory of minimal surfaces to prove local regularity by a blow up argument we want to see what is special of the "blow up limits" of our problem.

The main theorem in this regard is that global solutions are convex. More precisely

Theorem 3. ([C1]) *Let, as before, w be a normalized solution of our problem in $B_1(0)$. Then, there exists a universal modulus of continuity $\sigma(r)$ ($\sigma(0^+) = 0$), such that second pure derivatives of w*

$$D_{ee}w$$

satisfy

$$D_{ee}w(X) \geq -\sigma(|X|)\ .$$

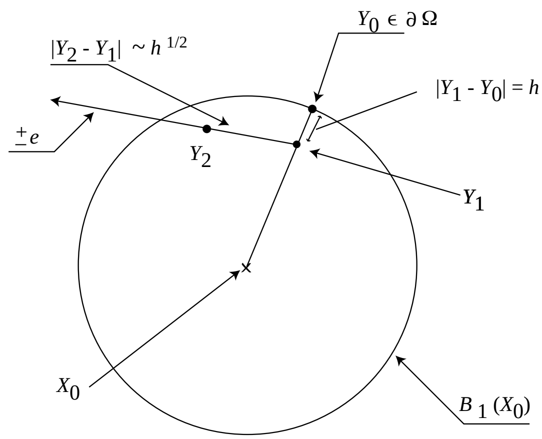
We prove the theorem through the following lemma applied inductively.

Lemma 7. *Assume that $B_r(X_0) \subset \Omega(w)$, and tangent to the free boundary, $\partial\Omega$, at a point Y_0.*

Let $-\alpha = \inf_{B_r(X_0)} D_{ee}w$.

Then $D_{ee}w(X_0) \geq -\alpha + C\alpha^M$ *for some C, M depending only on dimension.*

Proof of Lemma 7. By rescaling w to $\overline{w} = \frac{1}{r^2}w(rx)$, we may assume $r = 1$. (Note that this rescaling does not change bounds on second derivatives.



We do the following construction: In the ray (X_0, Y_0) choose a point, $Y_1 = X_0 + (1-h)(Y_0 - X_0)$, at distance h from Y_0. Since Y_0 belongs to $\partial\Omega$

$$w(Y_1) \leq Ch^2 \quad \text{and} \quad \nabla w(Y_1) \leq Ch \ .$$

Starting at Y_1, the direction $+e$ or $-e$ points "inwards" to the ball (i.e., $\langle \text{tor} -e, Y_0 - X_0 \rangle \leq 0$). Say e. (Note that $D_{ee} = D_{-e,-e}$.) Therefore if $Y_2 = Y_1 + \frac{1}{4} h^{1/2} e$, the segment from Y_1 to Y_2, remains at distance at least $h/4$ from ∂B_1 (for h small).

At Y_2, w is still positive. Let us show that this means that $D_{ee}w$ must be "almost positive" at some point in the segment $I = (Y_1, Y_2)$.

Indeed we represent $w(Y_2) - w(Y_1)$ as a double integral of D_{ee} along I

$$\begin{aligned} 0 \leq w(Y_2) &= w(Y_1) + \langle \nabla w(Y_1) \cdot Y_2 - Y_1 \rangle \\ &+ \iint_I D_{ee} w \leq h^2 + h \cdot \overbrace{|Y_2 - Y_1|}^{h^{1/2}} + \iint_I D_{ee} \end{aligned}$$

That is

$$\iint_I D_{ee} w \geq -Ch^2 - Ch^{3/2} \ .$$

But I has length $h^{1/2}$. Therefore

$$(h^{1/2})^2 \sup_I D_{ee} w \geq -Ch^2 - Ch^{3/2} \ ,$$

or

$$\sup_I D_{ee} w \geq -Ch^{1/2} \ .$$

We have, therefore, a point $Y_3 \in I$, with $D_{ee}w(Y_3) \geq -Ch^{1/2}$. We want now to choose h so that we have an actual gain over the previous bound:

$$D_{ee}w + \alpha \geq 0 \ .$$

Thus if we choose $-Ch^{1/2} = -\alpha/2$, i.e., $h = \left(\frac{\alpha}{2C}\right)^2$. We have that

$$D_{ee}w(Y_3) + \alpha \geq \alpha/2 \ .$$

We now apply Harnack inequality to the non-negative harmonic function

$$v(X) = D_{ee}w + \alpha \ .$$

It says that

$$D_{ee}w(X_0) + \alpha = v(X_0) \geq Cv(Y_3) \cdot (1 - |Y_3|)^{\bar{M}} = C\frac{\alpha}{2}(h)^{\bar{M}} = C\alpha^{1+2\bar{M}} \ .$$

The lemma is complete.

Corollary 5. *Let w be a normalized solution.*

If $D_{ee}w|_\Omega \geq -\alpha$, then $D_{ee}w|_{\Omega \cap B_{1/2}} \geq -\alpha + C\alpha^M$.

Proof. Let $X_0 \in \Omega \cap B_{1/2}$, and $B_r(X_0)$ be the largest ball in Ω containing X_0. Since $0 \in \partial\Omega$, $B_r(X_0) \subset B_1$ and the lemma applies.

Proof of Theorem. By induction we have that, for w a normalized solution, since $w|_{B_{1/2}}$ is $C^{1,1}$

$$D_{ee}w|_{B_{1/2}} \geq -C \equiv -\alpha_0 \ .$$

We apply the lemma inductively and we get

$$D_{ee}w|_{B_{2^{-k}}} \geq -\alpha_k$$

with

$$-\alpha_{k+1} \geq -\alpha_k + C\alpha_k^M \ .$$

This implies $\alpha_k \geq -k^{-\varepsilon}$ for some small ε, or $\sigma(r) = -|\log r|^{-\varepsilon}$.

Corollary 6. *Let w be a solution in B_M, i.e., $\bar{w} = \frac{1}{M^2} w(MX)$ is a normalized solution. Then*

$$D_{\alpha\alpha} w|_{B_1} \geq \sigma \left(\frac{1}{M} \right) .$$

Corollary 7. *Let w be a solution in R^n, then w is convex, and $\Lambda(w) = \{w = 0\}$ is convex.*

The material discussed from here on is new: both the regularity proof and the classification of singular points.

At this point let us pause for a moment and study what are the possible candidates for global solutions w:

a) $w = \frac{1}{2}(x^+)^2$ (in some systems of coordinates).

These are the "blow up" limits that we expect to get if we start from a point where $\partial\Omega$ is a smooth surface separating Ω from Λ.

b) $w = \sum_1^n \lambda_i \frac{x_i^2}{2}$ with $\lambda_i \geq 0$, $\sum \lambda_i = 1$, always in some system of coordinates.

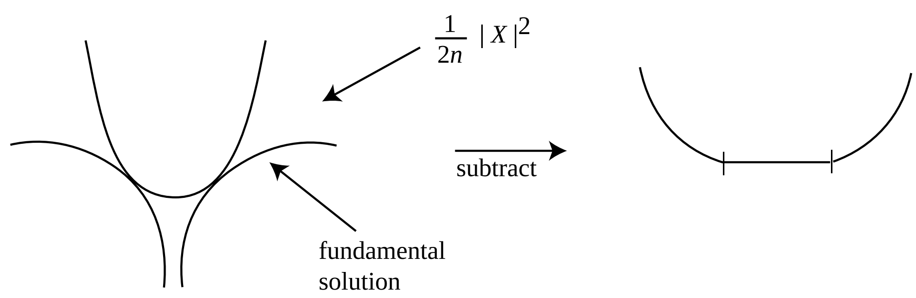
These are the solutions we expect if we started from a point where Λ had very little density or further an isolated point of Λ.

In fact:

Remark. If w is global and Λ a half space, w is as in a) from Cauchy-Kovalevski theorem.

If w is global and Λ has empty interior then w is as in b) from Liouville theorem.

c) We can construct a radial solution with Λ a ball:



So life is not so simple. But at least we can say that if w is a global solution, $0 \in \Lambda$ and the trace of Λ in say ∂B_1 is not "lower dimensional", then near the origin all level surfaces of w are smooth Lipschitz surfaces.

Let us write, from now on $X = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1})$.

Lemma 8. *Let w be a global solution (with $0 \in \partial\Omega$) and assume that $B_\rho(-te_n) \subset \Lambda$, for some $0 \le t \le 1/2$.*

Then a) for $|x'| \le \frac{\rho}{8}$, $-t < x_n < 1$, for any unit vector σ, with $\sigma_n > 0$, $|\sigma'| \le \rho/8$, we have

a$_1$) $D_\sigma w \ge 0$

a$_2$) All level surfaces, $w = \lambda$ are Lipschitz graphs.

$$x_n = f(x', \lambda) \quad \text{with} \quad \|f\|_{\mathrm{Lip}} \le \frac{C}{\rho} \ .$$

a$_3$) $D_{e_n} w(X) \ge C(\rho) d(X, \partial\Lambda)$.

a$_4$) For $|\sigma'| \le \rho/16$

Note. a$_2$), a$_3$) and a$_4$) follow from a$_1$).

Proof. Since w is convex, the directional derivative, $D_\sigma w$ is monotone along any line in the direction σ, and thus, it becomes positive once such a line intersects Λ.

This proves a$_1$) and thus a$_2$). This also proves that w grows quadratically in the vertical direction that is, if $X = (x', x_n)$, then $w(X) \ge C\rho^2 |x_n - f(x', 0)|^2$. Indeed, consider the ball

$$B = B_{\frac{1}{16}\rho|x_n - f|}(x', f(x', 0)) \ .$$

From maximal growth

$$w(Y) = \sup_B w \ge C\rho^2 |x_n - f|^2$$

but w is monotone increasing along the segment that joins Y to X, from a$_1$).

Finally, this implies a$_3$: Indeed

$$\int_{(x', f(x'))}^{(x', x_n)} D_{e_n} \, dy_n \ge C\rho^2 |x_n - f|^2 \ ,$$

therefore, $\sup_I D_{e_n} \geq C\rho^2|x_n - f|$. But this can only happen at a point along I, at distance $C\rho^2|x_n - f|$ from the free boundary.

From Harnack inequality, this also holds for all of the segment between say $C_1\rho^2|x_n - f|$ and $|x_n - f|$ with some large constant $C(\rho)$. (Notice that the factor $|x_n - f|$ scales out of the computation.)

Finally, a_4) follows by expressing any such direction σ, as

$$\sigma = a\tilde{\sigma} + be_n$$

with $\tilde{\sigma}$ a direction for which a_1) applies.

Then a, b can be chosen positive and $b \geq b_0 > 0$, b_0 a universal constant (1/8?). (Check this out)

I will now invoke the theory of harmonic functions in Lipschitz domains to deduce that all these level surfaces are uniformly $C^{1,\alpha}$ all the way to the free boundary.

Let me state the main theorems (see [CFMS], [J-K] and [A-C]).

Theorem 4. *Consider the domain $B_1^+ = B_1 \cap \{x_n > 0\}$. Let v_1, v_2 be two non-negative solutions of a divergence operator (a_{ij} bounded measurable)*

$$D_i a_{ij} D_j v = 0$$

and

$$v_i|_{x_n=0} \equiv 0 \ .$$

Let us normalize them so $v_1(\frac{1}{2}e_n) = v_2(\frac{1}{2}e_n) = 1$. Then, the quotient

$$u(X) = \frac{v_1(X)}{v_2(X)}$$

is bounded and Hölder continuous up to $x_n = 0$ in $B_{1/2}^+$, with $\|u\|_{L^\infty}, \|u\|_{C^\alpha} \leq C$ depending only on the ellipticity of a_{ij}.

Remark. The hypothesis of v_i are invariant under bilipschitz transformations of B_1^+. Indeed, we transfer the weak formulation of $D_i a_{ij} D_j v = 0$. That is

$$\int D_i \psi a_{ij} D_j v = 0$$

for any H_0^1 test function ψ or

$$\int (\nabla \psi)^T RA \nabla v \, dX = 0 \ .$$

Change variables $Y = Y(X)$. Then

$$\nabla_X \psi = \mathcal{T}_Y^X \nabla \psi(Y)$$

and

$$dX = \det \mathcal{T}_X^Y \, dY \ .$$

So

$$\int (\nabla_Y \psi)^T A^* \nabla_Y v \, dY = 0$$

for a bounded measurable A^*.

Corollary 8. *Same result holds if instead of B_1^+, we consider the domain*

$$D = \{|x'| \leq 1, f(x') \leq x \leq M\}$$

for some $f(x')$ Lipschitz, $|f(x')| \leq M/2$.

Application to our free boundary problem (still global solutions)**.**

Theorem 5. *Under the hypothesis of Theorem 4, the level surfaces $\{x_n = f(x', \lambda)\}$ are uniformly $C^{1,\alpha}$ up to $\lambda = 0$.*

Proof. We show that $\frac{w_\sigma}{w_n}$ are uniformly Hölder continuous in $\{|x'| \leq \frac{\rho}{8}, |x_n| \leq 1\}$ for any "horizontal" σ.

To apply the theorem we miss the positivity of w_σ. Write $\bar\sigma = \frac{\rho}{16}\sigma + e_n$. Then a$_4$) applies to $\bar\sigma$ (although it is not unitary) and according to Corollary 8, the quotient $\frac{w_{\bar\sigma}}{w_{e_n}}$ is C^α up to $\partial\Omega$.

That is $\frac{\rho}{16}\left(\frac{w_\sigma}{w_{e_n}}\right) + 1$ is C^α up to the free boundary and hence

$$u_\sigma = \frac{w_\sigma}{w_{e_n}}$$

is C^α up to the free boundary.

But u_σ is simply, $D_\sigma f(x', \lambda)$. Therefore, the graphs $x_n = f(x', \lambda)$, are uniformly $C^{1,\alpha}$ all the way up to $\lambda = 0$.

We next show, in order to complete the theory, that it is not necessary to pass to the limit, i.e., require w to be a global solution to reproduce the geometry above. This is a rather unusual fact, i.e., that in a finite approximation we can directly get regularity, characteristic of this particular problem. First an approximation lemma that says that we can almost reproduce the geometry of the global solutions if our w is a solution on a large enough ball.

Lemma 9. *Fix $\rho > 0$, $\varepsilon > 0$. Assume that we are given a solution w in a ball $B_M(0)$, with $M = M(\rho, \varepsilon)$ "very" large.*

Assume also that $w|_{B_1(0)}$ has the following property:

> *"The set $\Lambda(w) \cap B_1$ cannot be enclosed in any strip $\{\alpha < x_n < \beta\}$ of width, $(\beta - \alpha) \leq 4n\rho$"*

Then, for $M \geq M(\rho, \varepsilon)$ large there exist a global solution $w_\infty(X)$, that satisfies the hypothesis of Theorem 4, and such that

$$\|w - w_\infty\|_{L^\infty(B_1)} \quad , \quad \|\nabla w - \nabla w_\infty\|_{L^\infty(B_1)} \leq \varepsilon \ .$$

Proof. The proof is by compactness: Assume that such an $M(\rho, \varepsilon)$ does not exist. That means that if we fix (ρ, ε), we can find a sequence of solutions w_k defined in balls $B_k(0)$ that satisfy the hypothesis of the theorem and not the conclusion.

The w_k are a compact family in $C^{1,\alpha}$ in compact sets (they all vanish with its gradient at the origin and are universally $C^{1,1}$ in $B_{k/2}$. Thus a subsequence converges uniformly in compact sets to some function w_∞. To get a contradiction, it suffices to show that w is a global solution satisfying the hypothesis of Theorem 4. Indeed

a) $w_\infty \geq 0$, $w_\infty \in C^{1,1}$, and $\Delta w_\infty(x) = 1$, whenever $w_\infty(X) > 0$

b) $w_\infty(0) = 0$ and $0 \in \partial\Omega$ because $\sup_{B_r(0)} w_k \geq Cr^2$ independently of k and therefore $\sup_{B_r(0)} w_\infty \geq Cr^2$.

Also, w_∞ and $\Lambda(w)$ are convex and further:

"$\Lambda(w_\infty) \cap B_1$ cannot be enclosed in any strip of width $\frac{7}{2}n\rho$".

Indeed if $\Lambda(w_\infty)$ is enclosed in such a strip, say

$$\alpha < x_n < \beta\ , \qquad \beta - \alpha = \frac{7}{2}n\rho$$

then $w_\infty \geq \delta > 0$ outside $\{\alpha - \varepsilon \leq x_n \leq \beta + \varepsilon\} \cap B_1$. But w_k is converging uniformly to w in compact sets, thus $w_k \geq \frac{\delta}{2} > 0$ outside

$$\{\alpha - \varepsilon \leq x_n \leq \beta + \varepsilon\} \cap B_1$$

contradicting one of the hypothesis. To complete the proof of the theorem, we need to show that the "strip property" implies that $\Lambda(w_\infty) \cap B_1$ contains a ball of radius ρ.

For that we invoke a lemma of F. John that says that if E is the ellipsoid of largest volume contained in a convex set, in our case, $(\Lambda(w_\infty) \cap B_1)$ then $nE \supset (\Lambda(w_\infty) \cap B_1)$.

With this lemma at hand, if E has one of its diameters smaller than 2ρ, nE has one of its diameters smaller than $2n\rho$ and we can trap $\Lambda(w_\infty) \cap B_1$ in a $2n\rho$ strip, a contradiction.

In order to be able to jump now from the limiting configuration to an approximating one, we need the following curious property of the set $\Omega(w)$ for a normalized solution w.

Lemma 10. *Let h be a harmonic function in $\Omega(w)$. Assume*

a) *$h \geq 0$ on $\partial\Omega$ (for instance $\underline{\lim}\, h \geq 0$)*

b) *If N_σ denotes the σ neighborhood of $\partial\Omega$,*

b_1) $h|_{N_\sigma} \geq -\sigma$

b_2) $h|_{\Omega\setminus N_\sigma} \geq 1$

Then there exists a universal σ_0 such that for $\sigma < \sigma_0$, the hypothesis above imply that $h \geq 0$ in $\Omega \cap B_{1/2}$.

Proof. Suppose not, then there exists an X_0, in $N_\sigma \cap B_{1/2}$, where $h \leq 0$ (since outside of N_σ, $h \geq 1$). Consider, in $B_{1/4}(X_0)$, the function

$$v = h(X) - \delta\left[w(X) - \frac{1}{2n}|X - X_0|^2\right] .$$

Then (this must sound familiar!)

a) v is harmonic in $B_{1/4}(X_0) \cap \Omega$

b) $v(X_0) \leq 0$

Thus

c) v must have a negative minimum in $\partial(B_{1/4} \cap \Omega)$.

But along $\partial\Omega$, $v \geq 0$, thus the minimum occurs along $\partial B_{1/4}$.

Let us see that this is not possible. On $\partial B_{1/4} \cap N_\sigma$,

$$v \geq -\sigma - C\delta\sigma^2 + \frac{\delta}{2n}\left(\frac{1}{4}\right)^2 > 0$$

if $\delta \geq 200n\sigma$, and σ small. On $\partial B_{1/4} \cap (\Omega \setminus N_\sigma)$

$$v \geq 1 - C\delta^2 \geq 0$$

if δ (universally) small. □

As a corollary we get the following.

Theorem 6. *There exists an $\varepsilon = \varepsilon(\rho)$ such that*

If *w satisfies the conditions of Lemma 9, that is: w is a solution in a ball B_M,*

$M(\rho,\varepsilon(\rho)) = M(\rho)$ and $\Lambda(w)\cap B_1$ cannot be trapped in a strip of width $2n\rho$,

then *in an appropriate system of coordinates:*

a) *Conclusions of Lemma 8 hold for w (with $|x'| < \frac{\rho}{8}$ substituted by $|x'| < \frac{\rho}{16}$ and $|\sigma'| < \frac{\varphi}{16}$ substituted by $|\sigma'| < \frac{\rho}{32}$ in* a$_4$.

b) *Conclusions of Theorem 5 hold, that is all level surfaces $f(x',\lambda)$ are uniformly $C^{1,\alpha}$ up to $\lambda = 0$.*

Proof. From Lemma 9, we have that there exists a w_∞, satisfying the hypothesis of Lemma 8, such that $|w - w_\infty|_{L^\infty}, |\nabla w - \nabla w_\infty|_{L^\infty} \le \varepsilon$. Therefore, the crucial properties a$_3$), a$_4$) that make $f(x',\lambda)$ uniformly Lipschitz graphs, are "almost" satisfied. We will now use Lemma 9, to show that these properties are "fully" satisfied. We start by organizing the information we have, by putting Lemma 8 and Lemma 9 together.

Lemma 11. *Let w satisfy the hypothesis of Lemma 9, and let w_∞ be its ε global approximation. Then, in the domain $|x'| \le \frac{\rho}{8}$, $-t \le x_n \le 1$, we have ($N_\varepsilon(S)$ the ε-neighborhood of S)*

a) $\partial\Omega(w) \subset N_{\bar C(\rho)\sqrt{\varepsilon}} \quad (\partial\Omega(w_\infty))$

b) $D_{e_n} w(X) \ge C(\rho)[d(X,\partial\Lambda) - C\sqrt{\varepsilon}]$

c) *For $|\sigma'| \le \frac{\rho}{16}$*

$$D_\sigma w(X) \ge \bar C(\rho)[d(X,\partial\Lambda) - C\sqrt{\varepsilon}]$$

d) $w(X) \ge C(\rho)(d(X,\partial\Lambda) - C\sqrt{\varepsilon})^2$

Proof. All we have to prove is a) and then we just put the estimates in Lemmas 8 and 9 together.

To prove a) we note that if X_0 is in $\Omega(w_\infty)$ and $d(X_0,\partial\Omega(w_\infty)) \ge \bar C(\rho)\sqrt{\varepsilon}$ then $w_\infty(X_0) \ge C(\rho)(\bar C(\rho)\sqrt{\varepsilon}\,)^2 \ge 2\varepsilon$. Thus, $w(X_0) > 0$ and X_0 cannot belong to $\partial\Omega$.

On the other hand if $X_0 \in \bar\Omega(w)\cap\Lambda(w_\infty)$, from non-degeneracy,

$$\sup_{B_{\bar C(\rho)\sqrt{\varepsilon}}} w \ge C(\bar C(\rho)\sqrt{\varepsilon}\,)^2 \ge 2\varepsilon\ ,$$

thus $B_{\bar{C}(\rho)\sqrt{\varepsilon}}(X_0)$ cannot be contained in $\Lambda(w_\infty)$. $\square$

Now, to complete the proof of Theorem 6, all we have to prove that w satisfies a$_1$) of Lemma 8. (Now with $|\sigma'| \leq \frac{\rho}{16}$) since a$_2$), a$_3$) and a$_4$) follow from it.

From c) of Lemma 9,

$$D_\sigma w(X) \geq [\bar{C}(\rho) d(X, \partial\Lambda) - C\sqrt{\varepsilon}\,] \ .$$

We apply Lemma 9 to $D_\sigma w$ in $B_{\rho/8}$. Let $h = \frac{D_\sigma w}{\varepsilon^{1/4}}$.

Then if $d(X, \partial\Lambda) \geq \frac{2\varepsilon^{1/4}}{\bar{C}(\rho)}$

$$h(X) \geq 2 - C\varepsilon^{1/4} \geq 1 \ .$$

If $d(X, \partial\Lambda) \leq \frac{2\varepsilon^{1/4}}{\bar{C}(\rho)}$,

$$h(X) \geq -C\varepsilon^{1/4} \ .$$

If we choose ε small enough so that $C(\rho)\varepsilon^{1/4} < (\rho/8)\sigma_0$, we have $h(X) \geq 0$ in $B_{\rho/16}(0)$. Since outside of $B_{\rho/16}, D_\sigma$ is already non-negative, since we are away from Λ, the proof is complete.

By inverting the relation $M = \bar{M}(\rho)$ into $\rho = \rho(M)$, we have proven the following theorem.

Theorem 7. *Let w be a normalized solution.*

Then there is a universal modulus of continuity $\sigma(r)$. (more precisely $\sigma(r) = \rho(\frac{1}{r})$) such that if for one value of r, say r_0, $\Lambda(w) \cap B_{r_0}$ cannot be enclosed in a strip of width $r_0\sigma(r_0)$, then, in an r_0^2 neighborhood of the origin, the free boundary is a $C^{1,\alpha}$ surface $x_n = f(X')$ with

$$\|f\|_{C^{1,\alpha}} \leq \frac{C}{r_0} \ .$$

Proof. Let us renormalize w by r_0, i.e., consider

$$\bar{w} = \frac{1}{r_0^2} w(r_0, X) \ .$$

Then $\bar{w}$ is defined in a ball B_M of radius $M = \frac{1}{r_0}$ and $\Lambda(\bar{w}) \cap B_1$ cannot be enclosed in any strip of width $\sigma(r_0) = \rho(M)$. Thus, Lemma 9 applies.

The Structure of the set of singular points.

It only remains, now, to study the structure of the set of singular points of N, $\text{Sing}(w)$, that is those X_0, in $\partial\Omega$, for which $|\Lambda \cap B_r(X_0)| \subset S_{r\sigma(r)}$ (a strip of width $r\sigma(r)$) for every positive r.

Our main objective is to prove the following

Theorem 8. *Given a singular point, say $X_0 = 0 \in \partial\Omega$*

a) *There exists a unique* non-negative *quadratic polynomial*

$$Q_{X_0} = \frac{1}{2}(X^T M X)$$

with $\Delta Q_{X_0} = \text{trace } M = 1$, such that

$$|(w - Q_{X_0})(X)| \leq |X|^2 \sigma(|X|)$$

for some (universal) modulus of continuity $\sigma(|X|)$.

b) *$M(X_0)$ is continuous on X_0, for x_0 in* $\text{Sing}(w)$.

c) *If* $\dim \ker M = k$, *the singular set* $\text{Sing}(w)$, *lays in a neighborhood of X_0, in a k-dimensional C^1 manifold.*

The size of the neighborhood depends on the smallest non zero eigenvalue of M.

Before entering into the proof of this theorem, let us point out that, because of compactness, if X_0 is a singular point $w|_{B_r(X_0)}$ looks "more and more" as a quadratic polynomial:

Lemma 12. *Given ε, there exists an $M(\varepsilon)$, such that, if w is a solution in B_M and if 0 is a singular point of $\partial\Omega$, then, for some non-negative quadratic polynomial $Q = X^T A X$, with $\Delta Q = 1$, we have that*

$$|w - Q|_{B_1} \leq \varepsilon\ .$$

Proof. Suppose not, then there is a sequence of solutions $w_k(X)$ defined in $B_k(0)$, that have zero as a singular point, and for which there is no such polynomial.

Let us take a subsequence, w_k, that converges uniformly in compact sets to a global solution, w_∞. We prove: w_∞ is a quadratic polynomial Q as above.

Remark 1. Λ_∞ has empty interior. If not, Λ_∞ being convex, $\Lambda_\infty \cap B_1$ will also have nonempty interior, i.e., will contain a ball $B_{r_0}(X_0)$, where $w_\infty \equiv 0$. From nondegeneracy $w_k \equiv 0$ on $B_{r_0/2}(X_0)$ for k large. If not

$$\sup_{B_{\frac{1}{4}r_0}(Y_k)} w_k \geq C r_0^2 \quad \text{for any} \quad Y_k \in B_{r_0/2}(X_0) \cap \Omega(w_k) \ ,$$

contradicting the uniform convergence of w_k to w_0.

But, since 0 is a singular point of w_k, $\Lambda(w_k) \cap B_1$ must be contained in a strip of width $\sigma(\frac{1}{k})$ according to Theorem 7, a contradiction as soon as $\sigma(\frac{1}{k})$ becomes smaller than r_0.

Remark 2. w_∞ is a quadratic polynomial. Indeed $\Delta w_\infty \equiv 1$ and has quadratic growth, so

$$h = w_\infty - \frac{1}{2n}|X|^2$$

is globally harmonic with quadratic growth, thus h and hence w_∞ is a quadratic polynomial.

Corollary 9. *Let w be a normalized solution, and 0 a singular point. Then given ε, there exists an $\Gamma_0(\varepsilon)$ so that for any $r \leq r - 0$, we have a quadratic polynomial $Q^r = X^T A X$, with*

$$|w - Q^r|_{B_r} \leq \varepsilon r^2 \ .$$

As before, we invert the relation $\varepsilon(r) = \sigma(r)$ and say: If w is a normalized solution and zero is a singular point, there exists a Q_r such that

$$|w - Q^r|_{B_r} \leq r^2 \sigma(r) \ .$$

The problem with Lemma 11, is the standard problem in singularity theory, for instance in minimal surface theory: You take a sequence of blow ups of a minimal surfaces, and you get a minimal cone.

The problem is that different sequences may give different cones, that is the cone (or in our case the quadratic polynomial) may slowly rotate. The question is thus how to "glue" the polynomials Q_r that approximates a normalized solution at level B_r.

This is solved by the use of a *monotonicity formula*:

Theorem 9. ([ACF]) *In $B_1(0)$, let u_1, u_2 be two continuous functions such that*

a) *Have disjoint supports:* $u_1 \cdot u_2 = 0$

b) $u_1(0) = u_2(0) = 0$

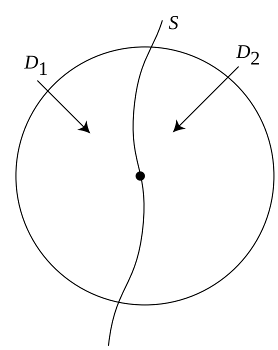
c) $u_i \Delta u_i > 0$

Then

$$\mathcal{T}(R) = \left(\frac{1}{R^2} \int_{B_R} \frac{(\nabla u_1)^2}{r^{n-2}} \, dX \right) \left(\frac{1}{R^2} \int_{B_R} \frac{(\nabla u_2)^2}{r^{n-2}} \, X \right) = \mathcal{T}_1 \cdot I_2$$

is monotone increasing in R.

Let me make several remarks about this theorem. The standard picture to understand this theorem, is to have in $B_1(0)$ a (relatively nice) surface S, through the origin, separating B_1 in two domains D_1 and D_2.



In each of them we have a harmonic function v_i that vanishes along S. Then a) each of the terms $\mathcal{T}_i$ can be understood as an average of $(\nabla v_i)^2$, i.e., we are dividing the volume integral in a domain of size $\sim R^n$ by a factor $R^2 r^{n-2} \sim R^n$.

In fact if v_1 is the positive part of a linear function, $v_1(X) = \alpha x_1^+$

$$\mathcal{T}_1(R) \equiv C(n)\alpha^2 \ .$$

with $C(n)$ a precise constant related to the volume of the unit ball of R^n.

b) $\mathcal{T}_i$ has linear scaling, i.e., if $\bar{u}(X) = \frac{1}{\lambda}u(\lambda X)$,

$$\mathcal{T}_i\left(\frac{R}{\lambda}, \bar{u}\right) = \mathcal{T}_i(R, u) \ .$$

c) If S is smooth at zero, i.e., $D_\nu v_i$ exists, then

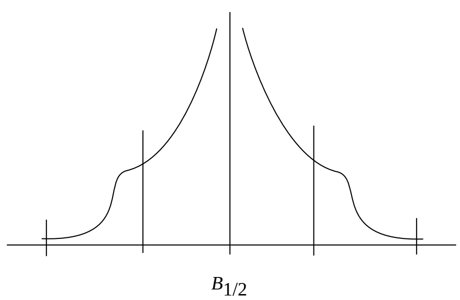
$$\lim_{R\to 0} \mathcal{T}_i = C(n)(D_\nu v_i)^2 \ .$$

In particular

$$C^2(n)(D_\nu v_1)^2(D_2 v_2)^2 \leq \mathcal{T}(1/2) \ .$$

d) On the other hand one may suspect that the quantity $\mathcal{T}_i(R)$ could be uncontrolled for some "not too bad" v_i. That is not so: Consider the function $\frac{1}{r^{n-2}}$ in $B_{1/2}$, and extended to a function V on B_1 in a smooth, non-negative way, so that $V \equiv 0$ near B_1.

Then



$$\begin{aligned}
\mathcal{T}_i(1/2) &= 2^2 \int_{B_{1/2}} \left(\frac{(\nabla v_i)^2}{r^{n-2}}\right) dX \\
&\leq 2^2 \int_{B_1} \Delta\left(\frac{v_i^2}{2}\right) \cdot V \, dX \\
&= 2^2 \int_{B_1 \setminus B_{1/2}} (\Delta V)\frac{v_i^2}{2} \, dX \leq C\|v_i\|^2_{L^2(B_1)}
\end{aligned}$$

We now go back to the classification theorem for singular points.

Proof of the main theorem.

Some previous remarks. Let 0 be a singular point of $\partial\Omega$. Recall that

a) $(\Lambda \cap B_r) \subset S_{r\sigma(r)}$ (a strip of width $r\sigma(r)$, and that

b) $\exists\, Q^r = \frac{1}{2}(X^T M^r X)$, with M non-negative, trace of $M = 1$, such that

$$|w - Q^r| \le r^2\sigma(r)$$
$$|\nabla w - Q^r| \le r\sigma^{1/2}(r)\ .$$

In rescaled terms, if $w^r = \frac{1}{r^2}w(rX)$ and $\Lambda^r = \Lambda(w^r)$,

$$|w^r - Q^r|_{B_1} \le \sigma(r)\ , \quad |\nabla(w^r - Q^r)| \le \sigma^{1/2}(r)$$
$$(\Lambda^r \cap B_1) \subset S_{\sigma(r)}\ .$$

In particular, outside of $S_{\sigma(r)^{1/3}}$, from a priori estimates:

$$|D_{ij}(w^r - Q^r)| \le \frac{1}{\sigma(r)^{2/3}}\,\mathrm{osc}(w^r - Q^r) \le \sigma(r)^{1/3}\ .$$

We are now ready for Step 1: Let $u_e = D_e w$, be a directional derivative of w. Then $\Delta u_e = 0$ in Ω and $u_e \equiv 0$ on Λ, so we can apply the monotonicity formula to u_e^+, u_e^-: The function

$$\mathcal{T}(R) = \left(\frac{1}{R^2}\int \frac{(\nabla u_e^+)^2}{r^{n-2}}\,dX\right)\left(\frac{1}{R^2}\int \frac{(\nabla u_e^-)^2}{r^{n-2}}\,dx\right) = \mathcal{T}^+(R,e)\mathcal{T}^-(R,e)$$

is monotone in R.

In Step 1, we will evaluate $\mathcal{T}^\pm(R,e)$ in terms of M^R (i.e., D^2Q^R)

$$\begin{aligned}\mathcal{T}^+(R,e) &= \frac{1}{R^2}\int_{\{(D_e w)>0\}\cap B_R} \frac{|D_{je}w|^2}{r^{n-2}}\,dX\\ &= \int_{\{D_e w^R>0\}\cap B_1} \frac{|D_{je}w^R|^2}{r^{n-2}}\,dX\ .\end{aligned}$$

We want to substitute $D_{je}w^R$ by $D_{je}Q^R = M^R_{je} = e_j^T M^R e$. From the general remarks above

$$\mathcal{T}^+(R,e) = \int_{D_e w^R > 0} \frac{\|M^R e\|^2}{r^{n-2}}\, dX + \sigma(r)^{1/3}$$

(we estimate the integral splittings in $S_{\sigma(R)^{1/3}}$ and $\mathcal{C}S_{\sigma(R)^{1/3}}$).

Next we want to substitute the domain

$$D = \{D_e w^R > 0\}\ , \quad \text{by } D^0 = \{D_e Q^R = X^T M^R e > 0\}\ .$$

If we denote by $D^{\pm} = \{D_e Q^R = X^T M^R e > \underset{(-)}{+}\, \sigma^{1/2}(R)\}$ we have, from the estimates above

$$D^+ \subset D \subset D^-\ .$$

Thus

$$\begin{aligned} &\|M^R e\|^2 \left(\int_{D^+} \frac{1}{r^{n-2}}\, dX\right) - \sigma(r)^{1/3} \\ &\qquad \le \mathcal{T}^+(R,e) \le \|M^R e\|^2 \left(\int_{D^-} \frac{1}{r^{n-2}}\, dX\right) + \sigma(r)^{1/3}\ . \end{aligned}$$

But $D^- \setminus D^+$ consists of a strip of width $\frac{\sigma(r)^{1/2}}{\|Me\|}$, hence

$$\Big| \int_{D^\pm} - \int_{D^0} \Big| \le C \min\left(\frac{\sigma(r)^{1/2}}{\|Me\|}, 1\right)\ .$$

Therefore, with

$$C^*(n) = \int_{B_1^+} \frac{dX}{r^{n-2}}$$

we have the final estimate

$$\begin{aligned} \mathcal{T}^+(R,e) &= \|M^R e\|^2 \left(C^* + 0\Big(\frac{\sigma(r)^{1/2}}{\|Me\|}\Big)\right) + 0(\sigma(r)^{1/3})) \\ &= C^* \|M^R e\| + 0(\sigma(r)^{1/3}) \quad \text{and} \\ \mathcal{T}(R,e) &= (C^*)^2 \|M^R e\|^2 + 0(\sigma(r)^{1/3})\ . \end{aligned}$$

(Remember that $\|M^R\|$ is universally bounded since $0 \le M^R$, and trace $M^R = 1$.

This completes Step 1.

In Step 2, we "glue" all of the M^R by the monotonicity formula.

Corollary of Step 1. *If $R_1 \le R_2$, for any vector e,*

$$\|M^R e\|^2 \le \|M^{R_2} e\|^2 + 0(\sigma^{1/3}(R_2)) \ .$$

Proof. It follows from the fact that $\mathcal{T}(R, e)$ is monotone in R.

Lemma 13. $\|M^{R_1} - M^{R_2}\| \le 0(\sigma^{1/6}(R_2))$.

Proof. Let $N = M^{R_2} - M^{R_1}$. Then N is symmetric, and trace $N = 0$. We have

$$\|(M^{R_2} - N)e\|^2 \le \|M^{R_2} e\|^2 + \sigma^{1/3}(R_2)$$

or

$$-2\langle M^{R_2} e, Ne\rangle + \|Ne\|^2 \le \sigma^{1/3}(R_2) \ .$$

Choose e the eigenvector, corresponding to $\lambda \le 0$ the smallest eigenvalue of N. Then

$$-2\lambda e^T M^{R_2} e + \lambda^2 \le 0^{1/3}(R_2) \ .$$

Since M^{R_2} is non-negative, we have $\lambda^2 \le \sigma^{1/3}(R_2)$ or $|\lambda| \le \sigma^{1/6}(R_2)$. Since $\operatorname{tr} N = 0$, $\|N\| \le \sigma^{1/6}(R_2)$.

Corollary 10.

a) *As R goes to zero M^R has a unique limit M^0, and*

b) $\|M^R - M_0\| \le \sigma^{1/6}(R)$

c) $\|w - X^T M_0 X\|\big|_{B_R} \le \|w - Q^R\| + \|Q^R - Q^0\| \le R^2\sigma^{1/6}(R)$.

d) *Let $Q^R_{X_0}$ denote the polynomial corresponding to center X_0, then for*

$$\begin{aligned}\|Q^R_{X_0} - Q^R_{X_1}\|_{B_R(X_0)\cap B_R(X_1)} &\le \|Q^R_{X_0} - w\| + \|Q^R_{X_1} - w\| \\ &\le R^2\sigma(R) \ .\end{aligned}$$

In particular if $\|X_1 - X_0\| \le \delta R$ for δ small

$$\|M^0_{X_0} - M^0_{X_1}\| \le \sigma(r) \ .$$

The rest of the theorem now follows.

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