REALIZING SETS OF ASSOCIATED PRIME IDEALS

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Introduction. Let R be a Noetherian ring and let S be a finite subset of $\operatorname{Spec} R$. It is shown in [ZS, Vol. I, Ch. IV, sec. 11, Theorem 21] that if no height 0 prime of R is contained in S, then there is an ideal I with S being exactly the set of associated prime ideals of I. However, that result is far from complete. We augment that argument to give a characterization of precisely which finite sets $S \subseteq \operatorname{Spec} R$ can be realized as the set of associated prime ideals of a given ideal, and use it to show that this occurs under a much weaker hypothesis than that of the result in [ZS]. For example, a corollary to our work shows that if no isolated primary component of 0 is prime, then for any finite $S \subseteq \operatorname{Spec} R$, there is an ideal I whose associated prime ideals comprise S.

Notation. R will be a commutative Noetherian ring. If I is an ideal of R, we will use $A(I)$ to denote the set of associated prime ideals of I. By $A(R)$ we will mean the empty set, so that we can allow the case $S = \emptyset$. We will use the phrase "primary decomposition" to automatically imply irredundancy (instead of using the more cumbersome "irredundant primary decomposition").

We will assume familiarity with basic results concerning primary decomposition. ([ZS, Vol. I, Ch. IV] provides adequate background.

Lemma 1. a) *Let $S \subseteq S'$. If there is an I' with $A(I') = S'$, then there is an I with $A(I) = S$.*

b) *If $S \subseteq A(0)$, then there is an I with $A(I) = S$.*

Proof. a) If $S = \emptyset$, we take $I = R$. Otherwise, say $A(I') = S' = \{P_1, \dots, P_m, P_{m+1}, \dots, P_n\}$ where $S = \{P_1, \dots, P_m\}$. Let $\bigcap_{i=1}^{n} p_i'$ be a primary decomposition of I' with p_i' primary to P_i for $1 \leq i \leq n$. Let $I = \bigcap_{i=1}^{m} p_i'$. Clearly this intersection is irredundant, as it is part of a longer irredundant intersection. Therefore, it is clear that $A(I) = \{P_1, \dots, P_m\} = S$.

b) This is immediate from (a). $\square$

In view of Lemma 1(b), we will not only consider cases with $S \not\subseteq A(0)$.

Lemma 2. *Let $A(0) = \{P_1, \dots, P_r, P_{r+1}, \dots, P_s, P_{s+1}, \dots, P_t\}$, and let $\bigcap_{i=1}^{t} p_i'$ be a primary decomposition of 0, with p_i' primary to P_i for $1 \leq i \leq t$. Also suppose that $p_1 \cap \cdots \cap p_r$*

is an irredundant intersection with p_i *primary to* P_i *for* $1 \leq i \leq r$*. Furthermore, suppose that* $Q_1, Q_2, \ldots, Q_w$ *are prime ideals outside of* $A(0)$*, and that for* $1 \leq k \leq w$*,* q_k *is* Q_k*-primary. Finally, suppose that for* $1 \leq i \leq r$*, no* P_i *contains any of* $P_{r+1}, \ldots, P_s$*,* $Q_1, \ldots, Q_w$*. Consider* $p_1 \cap \cdots \cap p_r \cap p'_{r+1} \cap \cdots \cap p'_s \cap q_1 \cap \cdots \cap q_w$*. Then none of the* p_i*,* $1 \leq i \leq r$*, and none of the* p'_i*,* $r+1 \leq i \leq s$*, can be deleted from this intersection without changing the value of the intersection. (That is, our intersection is at least partially irredundant, so far as its* p_i *and* p'_i *terms are concerned.)*

Proof. First assume that we could delete some p_i, with $1 \leq i \leq r$, from the intersection without changing the result. Without loss, we may assume $i = 1$. That would mean p_1 contains $(\bigcap_{i=2}^r p_i) \cap (\bigcap_{i=r+1}^s p'_i) \cap (\bigcap_{k=1}^w q_k)$, and so contains $(\bigcap_{i=2}^r p_i)(\prod_{i=r+1}^s p'_i)(\prod_{k=1}^w q_k)$. Since P_1 does not contain any of the p'_i, $r+1 \leq i \leq s$, or any of the q_k, $1 \leq k \leq w$, (because by hypothesis P_1 contains none of the radicals of those primary ideals), we see that p_1 (being P_1-primary) must contain $\bigcap_{i=2}^r p_i$. That contradicts the assumption that $p_1 \cap p_2 \cap \cdots \cap p_r$ is irredundant.

Now suppose that for some $r+1 \leq i \leq s$, we could delete p'_i from our intersection without changing the result. Without loss, suppose $i = r+1$. Then p'_{r+1} contains $(\bigcap_{i=1}^r p_i) \cap (\bigcap_{i=r+2}^s p'_i) \cap (\bigcap_{k=1}^w q_k)$, and so further intersecting with $(\bigcap_{i=1}^r p'_i) \cap (\bigcap_{i=s+1}^t p'_i)$, we see p'_{r+1} contains $(\bigcap_{i=1}^r p_i) \cap (\bigcap_{i=1}^r p'_i) \cap (\bigcap_{i=r+2}^s p'_i) \cap (\bigcap_{i=s+1}^t p'_i) \cap (\bigcap_{k=1}^w q_k)$. The middle three terms of this expanded intersection comprise every p'_i with $1 \leq i \leq t$ and $i \neq r+1$. Since the entire expanded intersection is contained in p'_{r+1}, we see that this expanded intersection is contained in $\bigcap_{i=1}^t p'_i$, which by assumption is 0. Thus, our expanded intersection is an intersection of primary ideals equaling 0. By deleting redundancies, this expanded intersection can be refined to a primary decomposition of 0. However, none of the radicals of any of the primary ideals in our expanded intersection equals P_{r+1}. That contradicts that $P_{r+1} \in A(0)$.

Notation. If S is a set of prime ideals of R, let $\min(S) = \{Q \in S \mid \text{there is no } Q' \in S \text{ with } Q' \subset Q\}$.

Proposition 3. *Let* R *be a Noetherian ring and let* S *be a finite subset of* $\operatorname{Spec} R$ *with* $S \not\subseteq A(0)$*. Let* $S_1 = S - A(0)$ *and* $S_2 = \{P \mid P \in S \cap A(0)$ *and there is no* $Q \in S_1$ *with* $Q \subseteq P\}$*.*

The following are equivalent.

a) *There is an ideal* I *with* $A(I) = S$*.*
b) *There is an ideal* K *with* $A(K) = \min(S_1) \cup S_2$*.*
c) *There is an ideal* J *with* $A(J) = S_2$ *such that* $J_Q \neq 0$ *for all* $Q \in \min(S_1)$*. (Here, if* $S_2 = \emptyset$*, then* $J = R$ *fulfills the requirement.)*

Proof. We first show that it will suffice to prove the equivalence of (a) and (c). Suppose that equivalence holds. Let $S' = \min(S_1) \cup S_2$, and note that $S'_1 = \min(S_1)$, so that $\min(S'_1) = \min(S_1)$. Also $S'_2 = S_2$. By the equivalence of (a) and (c) applied to S', we see that (b) holds if and only if (c) holds. Thus it will suffice to show (a) and (c) are equivalent. Let $S_3 = S - (S_1 \cup S_2)$, so that S is the disjoint union of S_1, S_2, and S_3. Furthermore, suppose $S_1 = \{Q_1, \ldots, Q_w\}$, $S_2 = \{P_1, \ldots, P_r\}$, and $S_3 = \{P_{r+1}, \ldots, P_s\}$.

a) $\Rightarrow$ c): Suppose there is an ideal I with $A(I) = S$. Consider some primary decomposition of I. For each $P_i \in S_2$ (i.e., $1 \leq i \leq r$), let the primary ideal which belongs to P_i and which appears in that primary decomposition of I be p_i. Let $J = p_1 \cap \cdots \cap p_r$. (Take $J = R$ if $S_2 = \emptyset$.) Clearly $A(J) = S_2$. Now consider some $Q \in \min(S_1)$. We must show $J_Q \neq 0$ (this being obvious if $J = R$). Let the primary ideal which belongs to Q and which appears in our primary decomposition of I be q. If $J_Q = 0$, a well known easy exercise shows there is an $x \in R - Q$ with $xJ = 0$. Thus $xJ \subseteq q$, from which we see that $J \subseteq q$. We now have $p_1 \cap \cdots \cap p_r \subseteq q$. This contradicts that $p_1, \ldots, p_r$, and q are all terms in our (irredundant) primary decomposition of I.

c) $\Rightarrow$ a): Suppose J is as in (c). As $A(J) = S_2$, consider a primary decomposition $J = \bigcap_{i=1}^{r} p_i$, with p_i primary to P_i for $1 \leq i \leq r$. As $S_2 \cup S_3 \subseteq A(0)$, let $A(0) = \{P_i \mid 1 \leq i \leq t\}$ (so $1 \leq r \leq s \leq t$), and let $\bigcap_{i=1}^{t} p'_i$ be a primary decomposition of 0, with p'_i primary to P_i. Let $I_0 = p_1 \cap \cdots \cap p_r \cap p'_{r+1} \cap \cdots \cap p'_s$. (Here, $\bigcap_{i=1}^{r} p_i$ means R if $S_2 = \emptyset$ and $\bigcap_{i=r+1}^{s} p'_i$ means R if $S_3 = \emptyset$.) We claim that $A(I_0) = S_2 \cup S_3$. The set of radicals of the primary ideals in the intersection defining I_0 comprise exactly $S_2 \cup S_3$. Therefore, to prove the claim, it will suffice to show that intersection is irredundant. We do that via Lemma 2 (applied to the case that there are no Q_k involved). This only requires that we show none of the $P_i \in S_2 = \{P_1, \ldots, P_r\}$ contain any $P_j \in S_3 = \{P_{r+1}, \ldots, P_s\}$.

We first note that if $P_j \in S_3$, then P_j must contain some $Q \in S_1$. If not, then we would have $P_j \in S_2$, which is a contradiction.

Suppose $P_i \in S_2$ and $P_j \in S_3$ and $P_j \subseteq P_i$. Then as the previous paragraph shows, for some $Q \in S_1$ we would have $Q \subseteq P_j \subseteq P_i$, which contradicts that $P_i \in S_2$. This completes the argument that $A(I_0) = S_2 \cup S_3$.

Recalling that $S_1 = \{Q_1, \ldots, Q_w\}$, we now assume that this numbering is such that for $1 \leq h < k \leq w$, $Q_k \nsubseteq Q_h$. We will inductively construct I_k, $1 \leq k \leq w$, such that $A(I_k) = S_2 \cup S_3 \cup \{Q_1, \ldots, Q_k\}$. Therefore, when $k = w$, we will have $A(I_w) = S_2 \cup S_3 \cup S_1 = S$, as desired.

Inductively assume that for $0 < k \leq w$, we have found primary ideals $q_1, q_2, \ldots, q_{k-1}$ belonging respectively to $Q_1, Q_2, \ldots, Q_{k-1}$, such that

$$I_{k-1} = \left(\bigcap_{i=1}^{r} p_i\right) \cap \left(\bigcap_{i=r+1}^{s} p'_i\right) \cap \left(\bigcap_{i=1}^{k-1} q_i\right) \quad \text{is irredundant}$$

(so that $A(I_{k-1}) = S_2 \cup S_3 \cup \{Q_1, \ldots, Q_{k-1}\}$).

We claim there is a Q_k-primary ideal q_k with $I_{k-1} \nsubseteq q_k$. To see this, first note that the Krull intersection theorem implies $\bigcap_{n=1}^{\infty} (Q_k)^n_{Q_k} = 0$ in R_{Q_k} [ZS, Vol. I, Ch. 4, sec. 7, Theorem 12, Corollary 12]. As the inverse image in R of $(Q_k)^n_{Q_k}$ is Q_k-primary, if our claim is false then I_{k-1} is contained in all those inverse images, showing $(I_{k-1})_{Q_k} = 0$. Therefore, to prove our claim, it will suffice to show $(I_{k-1})_{Q_k} \neq 0$. We consider two cases.

First, suppose $Q_k \in \min(S_1)$. Since a previous paragraph proved that every $P_j \in S_3$ contains some member of S_1, and since our Q_k is minimal in S_1, we see Q_k does not contain any $P_j \in S_3$. Considering the (inductively assumed) primary decomposition of I_{k-1}, that fact (and the minimality of Q_k in S_1) shows that $(I_{k-1})_{Q_k} = (p_1)_{Q_k} \cap \cdots \cap (p_r)_{Q_k} = (p_1 \cap \cdots \cap p_r)_{Q_k} = J_{Q_k}$. As $Q_k \in \min(S_1)$, the initial assumption on J shows

$(I_{k-1})_{Q_k} = J_{Q_k} \neq 0$. This proves the existence of q_k in the case that Q_k is a minimal member of S_1.

In the case that Q_k is not a minimal member of S_1, we may suppose $Q_h \in \min(S_1)$ and $Q_h \subset Q_k$. The ordering imposed on the primes in S_1 shows that we must have $h < k$. Since $I_{k-1} = (\bigcap_{i=1}^{r} p_i) \cap (\bigcap_{i=r+1}^{s} p'_i) \cap (\bigcap_{i=1}^{k-1} q_i)$ is a primary decomposition, localizing at Q_k shows

$$(I_{k-1})_{Q_k} = \left(\bigcap_{i=1}^{r} (p_i)_{Q_k}\right) \cap \left(\bigcap_{i=r+1}^{s} (p'_i)_{Q_k}\right) \cap \left(\bigcap_{i=1}^{k-1} (q_i)_{Q_k}\right)$$

will be a primary decomposition after the terms which equal R_{Q_k} are deleted. Since $Q_h \subset Q_k$, $(q_h)_{Q_k} \neq R_{Q_k}$, and so $(q_h)_{Q_k}$ will remain as part of a primary decomposition of $(I_{k-1})_{Q_k}$. If $(I_{k-1})_{Q_k} = 0$, this shows that in R_{Q_k}, $(Q_h)_{Q_k} \in A(0R_{Q_k})$. It follows that $Q_h \in A(0)$. This contradicts that $Q_h \in S_1$. We have now completed the proof that there is a Q_k-primary ideal q_k with $I_{k-1} \nsubseteq q_k$.

Let $I_i = I_{k-1} \cap q_k = (\bigcap_{i=1}^{r} p_i) \cap (\bigcap_{i=r+1}^{s} p'_i) \cap (\bigcap_{i=1}^{k} q_i)$. We claim this intersection is irredundant. Since each of $P_1, \ldots, P_r$ is in S_2, none of them contain any of $Q_1, \ldots, Q_k$. Also, we previously saw that none of $P_1, \ldots, P_r$ contain any of $P_{r+1}, \ldots, P_s$. Thus Lemma 2 shows that none of the p_i $(1 \leq i \leq r)$ or p'_i $(r+1 \leq i \leq s)$ is redundant in our intersection defining I_k. Of course q_k is also not redundant, since $I_{k-1} \nsubseteq q_k$. (If $k = 1$, the claim is now proven.) Suppose for some h with $1 \leq h \leq k-1$, q_h is redundant. Therefore q_h contains

$$\left(\bigcap_{i=1}^{r} p_i\right) \cap \left(\bigcap_{i=r+1}^{s} p'_i\right) \cap (q_1 \cap \cdots \cap q_{h-1} \cap q_{h+1} \cap \cdots \cap q_k)\ ,$$

and so q_h contains

$$q_k \left[\left(\bigcap_{i=1}^{r} p_i\right) \cap \left(\bigcap_{i=r+1}^{s} p'_i\right) \cap (q_1 \cap \cdots \cap q_{h-1} \cap q_{h+1} \cap \cdots \cap q_{k-1})\right]\ .$$

However, since $h < k$, our ordering of the elements in S_1 shows that $Q_k \nsubseteq Q_h$, and so $q_k \nsubseteq Q_h$. Thus q_h contains

$$\left(\bigcap_{i=1}^{r} p_i\right) \cap \left(\bigcap_{i=r+1}^{s} p'_i\right) \cap (q_1 \cap \cdots \cap q_{h-1} \cap q_{h+1} \cap \cdots \cap q_{k-1})\ .$$

However, that contradicts our inductive assumption that

$$\left(\bigcap_{i=1}^{r} p_i\right) \cap \left(\bigcap_{i=r+1}^{s} p'_i\right) \cap \left(\bigcap_{i=1}^{k-1} q_i\right)$$

is irredundant. This proves the claim that $(\bigcap_{i=1}^{r} p_i) \cap (\bigcap_{i=r+1}^{s} p'_i) \cap (\bigcap_{i=1}^{k} q_i)$ is irredundant, and so we see $A(I_k) = S_2 \cup S_3 \cup \{Q_1, \ldots, Q_k\}$ as desired.

The reader will see that the result in [ZS] mentioned in the introduction is a corollary of the next theorem (and Lemma 1(b) to cover the case $S \subseteq A(0)$).

Theorem 4. *Let R be a Noetherian ring, and let $S \subseteq \operatorname{Spec} R$ be finite. Suppose $S \not\subseteq A(0)$. Suppose for any prime Q minimal among members of $S - A(0)$ there is a prime $P \subseteq Q$ such that either $P \in A(0) - S$, or height $P = 0$ and the (unique) isolated component of 0 belonging to P is not P itself. Then there is an ideal I with $A(I) = S$.*

Proof. With notation as in Proposition 3, we must show there is an ideal J with $A(J) = S_2$ and $J_Q \neq 0$ for all $Q \in \min(S_1)$. As in the proof of Proposition 3, we let

$$A(0) = \{P_1, \ldots, P_r, P_{r+1}, \ldots, P_s, P_{s+1}, \ldots, P_t\} ,$$

and let $\bigcap_{i=1}^{t} p_i'$ be a primary decomposition of 0 with p_i' primary to P_i for $1 \leq i \leq t$. Since $S_2 = \{P_1, \ldots, P_r\}$, and since any height 0 prime P which happens to be in S must be contained in S_2, we may assume that $S_2 = \{P_1, \ldots, P_v, P_{v+1}, \ldots, P_r\}$ with $P_1, \ldots, P_v$ being the height 0 primes of R which are contained in S.

We let $J = P_1 \cap \cdots \cap P_v \cap p_{v+1}' \cap \cdots \cap p_r'$. This intersection is irredundant, (since $(\bigcap_{i=1}^{r} p_i'$ is and $P_1, \ldots, P_v$ have height 0). Therefore $A(J) = S_2$, as desired.

Now let Q be minimal in $S_1 = S - A(0)$. We must show $J_Q \neq 0$. Let $P \subseteq Q$ be as in the hypothesis. First suppose $P \in A(0) - S$. Then $P_Q \in A(0R_Q)$ but $P_Q \notin A(J_Q)$ (since $P \notin S_2 = A(J)$). Thus $J_Q \neq 0$.

Now suppose height $P = 0$ and the isolated component of 0 belonging to P is not P itself. If $P \notin S$, the previous case applies. Suppose $P \in S$. Then $P \in \{P_1, \ldots, P_v\}$. The definition of J shows P is an isolated component of J, and so P_Q is an isolated component of J_Q. If $J_Q = 0$, then P_Q is an isolated component of $0R_Q$, showing P is an isolated component of 0, which is a contradiction.

We now explore when it holds that for all finite S there is an I with $A(I) = S$.

Theorem 5. *Let R be Noetherian.*

a) *If no isolated primary component of* 0 *is prime, then for any finite $S \subseteq \operatorname{Spec} R$, there is an ideal I with $A(I) = S$.*

b) *The converse to part (a) is false. (However, the converse to part (a) is often true, as shown by part (d)(i) ⇒ (iii).)*

c) *The following are equivalent.*

  i) *For all finite $S \subseteq \operatorname{Spec} R$, there is an I with $A(I) = S$.*
  ii) *For all finite $S \subseteq \operatorname{Spec} R$ such that every prime in S has height at most* 1*, there is an I with $A(I) = S$.*

d) *Suppose every height* 0 *prime of R is contained in infinitely many height* 1 *primes. (For instance, this holds in the case that* $\operatorname{Dim} R/P \geq 2$ *for all height* 0 *primes P, using* [K, Theorem 144]*.) Then the following are equivalent.*

  i) *For any finite $S \subseteq \operatorname{Spec} R$ there is an ideal I with $A(I) = S$.*
  ii) *For all pairs of primes $P \subset Q$ such that height $P = 0$ and height $Q = 1$, and such that P is the only prime ideal properly contained in Q, there is an ideal I with $A(I) = \{P, Q\}$.*
  iii) *No isolated primary component of* 0 *is prime.*

e) *Let X be an indeterminate. Then for all finite $S \subseteq \operatorname{Spec} R[X]$ there is an ideal J of $R[X]$ with $A(J) = S$ if and only if in R, no isolated component of* 0 *is prime.*

f) *Let X, Y be indeterminates. If for all finite $S \subseteq \operatorname{Spec} R[X]$ there is an ideal J of $R[X]$ with $A(J) = S$, then for all finite $S \subseteq \operatorname{Spec} R$ there is an ideal I of R with $A(I) = S$ and for all finite $S \subseteq \operatorname{Spec} R[X, Y]$ there is an ideal K of $R[X, Y]$ with $A(K) = S$.*

g) *Let X be an indeterminate. If for all finite $S \subseteq \operatorname{Spec} R$ there is an ideal I of R with $A(I) = S$, it does not follow that the analogous statement is true for $R[X]$.*

Proof. a) If no isolated component of 0 is prime, then Theorem 4 shows that for any finite S with $S \not\subseteq A(0)$, there is an ideal I with $A(I) = S$, and Lemma 1(a) gives the same conclusion when $S \subseteq A(0)$.

b) Let $T = K[X, Y]/(X^2Y)$. With x and y the images of X and Y, let $P_1 = (x)$, $P_2 = (y)$, and $Q = (x, y)$. Now let $R = T_Q$. $\operatorname{Spec} R = \{P_{1Q}, P_{2Q}, Q_Q\}$ and $A(0) = \{P_{1Q}, P_{2Q}\}$. If $S \subseteq A(0)$ Lemma 1(b) shows there is an I with $A(I) = S$. Suppose $S \not\subseteq A(0)$. Then $S - A(0) = \{Q_Q\}$. Since $P_{1Q} \subseteq Q_Q$ and P_{1Q} is not an isolated component of 0, Theorem 4 shows there is an I with $A(I) = S$. Thus R satisfies the conclusion of part (a), but does not satisfy its hypothesis, since P_{2Q} is an isolated component of 0. (Note that P_{2Q} is not contained in infinitely many height 1 primes, and so R fails to satisfy the hypothesis of part (d).)

c) i) $\Rightarrow$ ii) is trivial. Suppose (ii) holds, and let S be an arbitrary finite set of prime ideals. If $S \subseteq A(0)$, then Lemma 1(b) gives (i). Now assume $S \not\subseteq A(0)$. We will first (possibly) add more prime ideals to S. Consider any $Q \in S$. If height $Q > 1$, then it easily follows from [K, Theorem 144] that there are infinitely many height 1 primes contained in Q, and so some of them are not in $A(0)$. In that case, adjoin to S a height 1 prime Q' with $Q' \subset Q$ and $Q' \notin A(0)$. Doing this for all $Q \in S$ with height $Q > 1$, we have increased S to a larger finite set S', in such a way that (with notation as in Proposition 3) $\min(S'_1)$ consists of height 1 primes, and S'_2 consists of primes of height 0 or 1. Let $S'' = \min(S'_1) \cup S'_2$. By (ii), there is an ideal I'' with $A(I'') = S''$. By Proposition 3(b) $\Rightarrow$ (a), there is is an ideal I' with $A(I') = S'$. Since $S \subseteq S'$, Lemma 1(a) shows there is an ideal I with $A(I) = S$.

d) Suppose every height 0 prime is contained in infinitely many height 1 primes. Clearly (i) $\Rightarrow$ (ii) is trivial and (iii) $\Rightarrow$ (i) is by part (a). Suppose (ii) holds but (iii) fails, so that for some height 0 prime P we have that the isolated component of 0 belonging to P is P itself. (We will derive a contradiction.) We may suppose that the height 0 primes of R are $P = P_1, P_2, \ldots, P_v$. Let Q be a height 1 prime which contains P. If Q contains $P_2 \cap \cdots \cap P_v$, then Q is minimal over $P + (P_2 \cap \cdots \cap P_v)$. As only finitely many primes are minimal over that ideal, our supposition shows there are infinitely many height 1 primes Q which contain P but do not contain any of $P_2, \ldots, P_v$. For such a Q, it is clear that P is the only prime properly contained in Q. By (ii), there is an I with $A(I) = \{P, Q\}$, so that $A(I_Q) = \{P_Q, Q_Q\}$. However, as $A(0)$ is finite and we have infinitely many choices for our Q, we may assume $Q \notin A(0)$. Then P is the only prime divisor of 0 contained in Q, and since P is an isolated component of 0, we see that $P_Q = 0$. As $I \subseteq P$, $I_Q = 0$. Thus in R_Q, $Q_Q \in A(I_Q) = A(0R_Q)$, showing that in R, $Q \in A(0)$, which is a contradiction.

e), f), and g): These are all easy exercises, using part (d)(i) $\Rightarrow$ (iii) and part (a), along with the following two facts.

(i) In $R[X]$, every height 0 prime is contained in infinitely many height 1 primes [K, sec.1–5].

(ii) Condition d(iii) holds in R if and only if it holds in $R[X]$ (which follows easily from the fact that a height 0 prime P of R is an isolated component of 0 if and only if there is an $r \in R - P$ with $rP = 0$).

For (g), we use the ring R constructed in part (b). We leave the details to the reader. □

The converse to Theorem 4 is false. Thus it is possible to have $S \nsubseteq A(0)$ and an ideal I with $A(I) = S$, but yet have a minimal member Q of $S - A(0)$ such that every prime $P \subseteq Q$ with $P \in A(0)$ has $P \in S$, and every height 0 prime $P \subseteq Q$ appears in the primary decomposition of 0. The ring R_1 below is such an example. It will be useful to introduce a definition.

Definition. Say $A(0)$ is critical if $A(H) = A(0)$ implies $H = 0$.

Examples 6. Let K be a field, $R_1 = K[X, Y, Z]/(X^3, XY)$, and $R_2 = K[X, Y, Z]/(X^2, XY)$. In both rings let x, y, and z be the images of X, Y, and Z. Then in both rings, $A(0) = \{(x), (x, y)\}$. Let $S = A(0) \cup \{(x, y, z)\}$. We claim that in R_1 there is an ideal I with $A(I) = S$, but in R_2 no such I exists. With notation as in Proposition 3, we have $S_1 = \{(x, y, z)\}$, $S_2 = A(0)$, and $S_3 = \emptyset$. By Proposition 3(c) $\Leftrightarrow$ (a), it will suffice to show that in R_1, there is an ideal J with $A(J) = S_2 = A(0)$ such that $J_{(x,y,z)} \neq 0$ while in R_2, no such J exists.

In R_1, take $J = (x) \cap (x^2, y)$. Clearly $A(J) = \{(x), (x, y)\} = A(0)$. As $x^2 \in J$ and as $\operatorname{Ann} x^2 = (x, y)$, we see $J_{(x,y,z)} \neq 0$. (Since $J \neq 0$ but $A(J) = A(0)$, we see $A(0)$ is not critical.)

To show that no such J exists in R_2, it will suffice to show that if $A(J) = A(0)$, then $J = 0$ (i.e., we want $A(0)$ to be critical). Thus suppose $A(J) = A(0)$, and let $J = p \cap q$ be a primary decomposition with p and q primary to (x) and (x, y) respectively. Since $p \nsubseteq q$, we have $(x) \nsubseteq q$. Since $J \subseteq (x) \cap q$, it will suffice to show that $(x) \cap q = 0$ whenever q is an (x, y)-primary ideal with $(x) \nsubseteq q$. Suppose $xf \in (x) \cap q$. Since $x \notin q$, we must have $f \in (x, y)$, showing $xf \in x(x, y) = 0$ (in R_2).

In the ring R_2 of Example 6, the fact that $A(0)$ is critical was influential. We explore that influence in more generality. (R_2 is an example of what the next proposition tells us.)

Proposition 7. *Let R be Noetherian. Suppose $A(0)$ is critical, and let S be a finite subset of $\operatorname{Spec} R$ with $S \nsubseteq A(0)$. Then there is an ideal I with $A(I) = S$ if and only if for every Q minimal in $S - A(0)$ there is a $P \subseteq Q$ with $P \in A(0) - S$.*

Proof. If for every Q minimal in $S - A(0)$ there is a $P \subseteq Q$ with $P \in A(0) - S$, then Theorem 4 shows there is an I with $A(I) = S$.

Conversely, suppose $A(I) = S$. Then by Proposition 3(a) $\Rightarrow$ (c), there is a J with $A(J) = S_2$ and $J_Q \neq 0$ for all Q minimal in S_1. Let Q be minimal in $S - A(0) = S_1$. We must show there is a prime $P \subseteq Q$ with $P \in A(0) - S$. Suppose that is false. That is, suppose $\{P \mid P \subseteq Q \text{ and } P \in A(0)\} = \{P \mid P \subseteq Q \text{ and } P \in A(0) \cap S\}$.

Since $A(0) \cap S = S_2 \cup S_3$, we have $\{P \mid P \subseteq Q$ and $P \in A(0)\} = \{P \mid P \subseteq Q$ and $P \in S_2 \cup S_3\}$. However, the second paragraph in the proof of Proposition 3(c) $\Rightarrow$ (a) shows that every prime contained in S_3 must contain a prime in S_1. Since Q is minimal in S_1, we therefore see that Q cannot contain any prime in S_3. Therefore, we have $\{P \mid P \subseteq Q$ and $P \in A(0)\} = \{P \mid P \subseteq Q$ and $P \in S_2\}$. It follows that $A(0R_Q) = \{P_Q \mid P \subseteq Q$ and $P \in S_2\}$. Recalling that $A(J) = S_2$, we also have $A(J_Q) = \{P_Q \mid P \subseteq Q$ and $P \in S_2\} = A(0R_Q)$. Since $J_Q \neq 0$, we have that $A(0R_Q)$ is not critical. In view of the next lemma, this contradicts that $A(0)$ is critical.

Lemma 8. *Let R be Noetherian, and let T be a set of prime ideals of R such that for all $Q' \in A(0)$ there is a $Q \in T$ with $Q' \subseteq Q$. The following are equivalent.*

i) $A(0)$ *is critical.*
ii) $A(0R_Q)$ *is critical for all* $Q \in T$.
iii) $A(0R_M)$ *is critical for all maximal ideals* M.
iv) $A(0R_Q)$ *is critical for all* $Q \in A(0)$.
v) $A(0R_Q)$ *is critical for all prime ideals* Q.

Proof. It will suffice to show that (i) and (ii) are equivalent, since (iii), (iv), and (v) are just special cases of (ii) for appropriate choices of T.

ii) $\Rightarrow$ i): Suppose $A(0R_Q)$ is critical for all $Q \in T$. Suppose $A(H) = A(0)$. (We want $H = 0$.) Let $Q' \in A(0)$ and suppose (by the hypothesis) that $Q \in T$ with $Q' \subseteq Q$. We have $A(H_Q) = \{P_Q \mid P \in A(H)$ and $P \subseteq Q\} = \{P_Q \mid P \in A(0)$ and $P \subseteq Q\} = A(0R_Q)$. Thus (by ii)) H_Q must be $0R_Q$, and so an element outside of Q kills H. Thus an element outside of Q' kills H. As that holds for all $Q' \in A(0)$, we see $H = 0$.

i) $\Rightarrow$ ii): Suppose $A(0)$ is critical but for some $Q \in T$, $A(0R_Q)$ is not. Let $A(0) = \{P_1, \dots, P_g, P_{g+1}, \dots, P_t\}$ with $P_i \subseteq Q$ exactly when $1 \leq i \leq g$ (so that $A(0R_Q) = \{P_{1Q}, \dots, P_{gQ}\}$). Let $\bigcap_{i=1}^t p'_i$ be a primary decomposition of 0. As $A(0R_Q)$ is not critical, there is an ideal J in R such that $J_Q \neq 0R_Q$, but $A(J_Q) = A(0R_Q)$. The latter shows that $\{P \mid P \in A(J)$ and $P \subseteq Q\} = \{P_1, \dots, P_g\}$. Fix some primary decomposition of J, and for $1 \leq i \leq g$, let p''_i be the primary component in that decomposition which belongs to P_i. (Thus $\bigcap_{i=1}^g p''_i$ is part of a primary decomposition of J.) Let $H = p''_1 \cap \cdots \cap p''_g \cap p'_{g+1} \cap \cdots \cap p'_t$. We see that $H_Q = \bigcap_{i=1}^g (p''_i)_Q = J_Q \neq 0R_Q$, and so $H \neq 0$.

We will now contradict that $A(0)$ is critical by showing $A(H) = A(0)$. For that, it will suffice to show that the above intersection of primary ideals defining H is irredundant. Suppose to the contrary that one of its terms can be deleted without affecting the result.

First suppose one of the p''_i for $1 \leq i \leq g$, and be deleted. Without loss, we may assume p''_1 can be deleted. Then p''_1 contains the intersection of the other terms, and so contains $(\bigcap_{i=2}^g p''_i)(\prod_{i=g+1}^t p'_i)$. As Q contains P_1 but does not contain any of $P_{g+1}, \dots, P_t$ P_1 does not contain any of $P_{g+1}, \dots, P_t$, and so P_1 does not contain $\prod_{i=g+1}^t p'_i$. As p''_1 is primary to P_1, we now see that p''_1 contains $(\bigcap_{i=2}^g p''_i)$. However, this contradicts that $\bigcap_{i=1}^g p''_i$ is part of a primary decomposition of J.

Next suppose we can delete one of p'_i, $g+1 \leq i \leq t$, from the intersection defining H, without affecting the result. Without loss, we may assume we can delete p'_{g+1}. Then p'_{g+1} contains $(\bigcap_{i=1}^g p''_i) \cap (\bigcap_{i=g+2}^t p'_i)$. We claim that $\bigcap_{i=1}^g p''_i$ contains $\bigcap_{i=1}^g p'_i$. If true, then we see p'_{g+1} contains $(\bigcap_{i=1}^{g} p'_i) \cap (\bigcap_{i=g+2}^{t} p'_i)$, which contradicts that $\bigcap_{i=1}^{t} p'_i$ is a primary decomposition of 0.

It only remains to prove the claim that $\bigcap_{i=1}^{g} p''_i$ contains $\bigcap_{i=1}^{g} p'_i$. Suppose x is contained in the latter intersection. Since that intersection is part of a primary decomposition of 0, and since the associated prime ideals of 0 which are contained in Q are exactly $P_1, \ldots, P_g$, we see that $(\bigcap_{i=1}^{g} p'_i)_Q = 0R_Q$. Thus there is a $y \in R - Q$ with $yx = 0$. Since for $1 \leq i \leq g$, we have $yx \in p''_i$ but $y \notin P_i$, (because $P_i \subseteq Q$) and since p''_i is primary to P_i, we see that $x \in \bigcap_{i=1}^{g} p''_i$ as desired.

Corollary 9. *Let R be Noetherian, and suppose S is finite with $A(0) \subseteq S \subseteq \operatorname{Spec} R$. If $A(0)$ is critical, then there is an ideal I with $A(I) = S$ if and only if $S = A(0)$.*

Proof. One direction is obvious. The other follows from Proposition 7. ☐

Corollary 10. *Let R be Noetherian and reduced. Then there is an ideal I with $A(I) = S$ if and only if for all $Q \in S - A(0)$ there is a height 0 prime $P \subseteq Q$ with $P \notin S$.*

Proof. As R is reduced, $A(0)$ is critical and consists exactly of the height 0 primes. The result follows from Proposition 7 and Lemma 1(b).

Remark 11. a) If R is Noetherian and $A(0)$ is critical, then every height 0 prime P is an isolated component of 0, since otherwise we would have $P_P \neq 0R_P$ but $A(0R_P) = \{P_P\} = A(P_P)$, contradicting Lemma 8(i) ⇒ (iv).

b) It is possible to have every height 0 prime be an isolated component of 0 without $A(0)$ being critical. The ring R_1 of Example 6 shows this, since $A(0) = \{(x), (x, y)\} = A((x^2, xy))$ but $(x^2, xy) \neq 0$, while the only isolated component of 0 is prime, since a primary decomposition of 0 is $(x) \cap (y)$.

c) The hypothesis of Proposition 7 cannot be weakened from saying $A(0)$ is critical to only saying every height 0 prime is an isolated component of 0, since the ring R_1 of Example 6 is a counter-example.

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