Emerging Scholars Program – Fall 2007

M210E – Calculus Workshop

Practice Exam 1 – Gold Version

Instructions: This exam contains six problems. Problems 1 through 4 are worth 15 points each; problems 5 and 6 are worth 20 points each. You have 75 minutes to work on this test. No calculators, books, notes, or other external aids are allowed. The group responsible for each problem is indicated at the beginning of the problem; some problems have been edited and/or modified.

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1. (Euler) Define a sequence $\{a_n\}_{n=1}^{\infty}$ by

$$a_n = \left(\frac{n+1}{n}\right)^{n/2}.$$

Compute the limit $\lim_{n\to\infty} a_n$. Your calculation should be as thorough as possible; if you want to use a fact about a particular limit that you have memorized, you should prove it.

2. (Gauss) Consider the improper integral

$$\int_0^\infty \frac{dx}{1+x^2}.$$

(a) Show that this integral converges, and compute its value.

(b) By the Integral Test, the convergence of this integral allows us to show that a certain series converges. Give this series, and carefully show that the hypotheses of the Integral Test hold (*i.e.* that the Integral Test actually applies).

(c) Use a method other than the Integral Test to show that the series you gave in (b) converges.

3. (Leibniz, Newton) For each of the following series, determine, with proof, whether the given series converges.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2 + 5n}{\sqrt{9 + n^7}}$$

(b)

$$\sum_{n=1}^{\infty} \frac{5^n + 2^n}{n!}$$

4. (Fourier) Consider the series

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n+1}\right).$$

(a) Show that this series satisfies the hypotheses of the Alternating Series Test, and therefore converges.

(b) Does this series converge absolutely? Justify your answer.

5. **(Lagrange)** Answer the two following questions.

(a) Find all real values of x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

(b) Find all real values of x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{(3n)!}$$

6. (Hilbert) Consider the series

$$\sum_{n=1}^{\infty} \big(\log(2n-1) - \log(2n)\big).$$

Recall that, for all real numbers $x > 0$, we have $\log x \leq x - 1$. You may use this inequality in this problem without proving it if you wish.

(a) Rewrite the terms of this series so that each term is expressed as a single logarithm, rather than a difference of two logarithms.

(b) Prove that the series above diverges.

(c) Define a sequence $\{a_n\}_{n=1}^{\infty}$ by

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$$

for all natural numbers n. Compute the limit $\lim_{n\to\infty} a_n$.