Emerging Scholars Program – Fall 2007

M210E – Calculus Workshop

Practice Exam 2 – Gold Version

Instructions: This exam contains six problems. Problems 1 through 4 are worth 15 points each; problems 5 and 6 are worth 20 points each. You have 75 minutes to work on this test. No calculators, books, notes, or other external aids are allowed. The group responsible for each problem is indicated at the beginning of the problem; some problems have been edited and/or modified.

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1. (Hilbert) Consider the function $f(x) = e^x$.

(a) Use a third-order Taylor series approximation (that is, the Taylor series through the x^3 term), centered at $c = 0$, to estimate the value of $e^{0.3}$.

(b) Give the Lagrange form of the remainder of the Taylor series you computed above. Based on this form, what is the greatest possible error in your approximation? (Don't worry about simplifying your answer here; just express it in a form such that you could enter it into your calculator and get a numerical value.)

(c) Give the Cauchy (integral) form of the remainder of the Taylor series you computed above. Don't worry about evaluating the integral.

2. (Fourier) Let $\mathbf{x}$ be the vector $\langle 3, 1, -2\rangle$, and let $\mathbf{y}$ be the vector $\langle -1, 0, 2\rangle$.

(a) Compute the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. (You may write your answer in terms of an inverse trig function if necessary.) Is this angle acute, right, or obtuse?

(b) Compute the scalar and vector projections of $\mathbf{x}$ onto $\mathbf{y}$.

3. (Fermat) Let P, Q, and R be the points $(1, -3, -2)$, $(2, 0, -4)$, and $(6, -2, -5)$, respectively.

(a) Determine whether the triangle ΔPQR is a right triangle.

(b) Compute the perimeter of ΔPQR.

(c) Compute the area of ΔPQR.

4. (Hilbert) Show that the following limits do not exist:

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy + y^3}{x^2 + y^2}$$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{x\cos y - x}{x^2 + \sin^4 y}$$

5. (Riemann) A particle travels through $\mathbb{R}^3$ with position function

$$\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle.$$

(a) Sketch the path of this particle in $\mathbb{R}^3$. Indicate in which direction the particle is traveling.

(b) Compute the velocity and acceleration functions of the particle. Sketch the velocity vector at time $t = \pi$ in your picture above.

(c) Find the unit tangent vector and the principal unit normal vector to the curve $\mathbf{r}(t)$ at time t.

(d) Using the method you learned in class, compute the length of the path of the particle from time $t = 0$ to time $t = \pi$.

6. (Pascal) Consider the planes given by the equations $x + y + z = 1$ and $x - 2y + 3z = 1$.

(a) Find the symmetric equation of the line that lies in both of these planes.

(b) Compute the distance from the point $(2, 0, 1)$ to this line.

Solutions

1. **(a)** The third-order Taylor approximation is

$$\begin{aligned}\sum_{n=0}^{3}\frac{f^{(n)}(c)}{n!}\cdot(x-c)^n &= \sum_{n=0}^{3}\frac{e^0}{n!}\cdot(0.3)^n\\ &= \frac{1}{0!}+\frac{1}{1!}\cdot(0.3)+\frac{1}{2!}\cdot(0.3)^2+\frac{1}{3!}\cdot(0.3)^3\\ &= 1+0.3+0.045+0.0045\\ &= 1.3495.\end{aligned}$$

(b) The theorem of Lagrange states that there exists some number z_3 in the interval $[0, 0.3]$ such that

$$R=\frac{f^{(4)}(z_3)}{4!}\cdot(0.3)^4=\frac{e^{z_3}}{24}\cdot(0.3)^4.$$

The maximum value of e^{z_3} on the interval $[0, 0.3]$ is $e^{0.3}$, so the error R satisfies the inequality

$$R\le\frac{e^{0.3}}{24}\cdot(0.3)^4.$$

(c) The Cauchy (integral) form of the remainder is

$$\begin{aligned}R &= \frac{1}{3!}\int_c^x f^{(4)}(t)(x-t)^3dt\\ &= \frac{1}{6}\int_0^{0.3}e^t(0.3-t)^3dt.\end{aligned}$$

2. **(a)** Let θ be the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. We have

$$\begin{aligned}\cos\theta &= \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||\cdot||\mathbf{y}||}\\ &= \frac{\langle 3,1,-2\rangle\cdot\langle -1,0,2\rangle}{||\langle 3,1,-2\rangle||\cdot||\langle -1,0,2\rangle||}\\ &= \frac{3\cdot-1+1\cdot0+-2\cdot2}{\sqrt{3^2+1^2+(-2)^2}\cdot\sqrt{(-1)^2+0^2+2^2}}\\ &= \frac{-7}{\sqrt{14}\cdot\sqrt{5}}\\ &= \frac{-7}{\sqrt{70}}.\end{aligned}$$

So $\theta=\cos^{-1}\left(-\frac{7}{\sqrt{70}}\right)$. This is the inverse cosine of a negative number, and therefore is obtuse.

(b) The scalar projection of $\mathbf{x}$ onto $\mathbf{y}$ is equal to

$$\frac{||\mathbf{x}||\cos\theta}{||\mathbf{y}||}=\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{y}||^2}=\frac{-7}{5}.$$

So the vector projection of $\mathbf{x}$ onto $\mathbf{y}$ is equal to

$$\mathrm{proj}_{\mathbf{y}}\mathbf{x}=\frac{-7}{5}\mathbf{y}=\frac{-7}{5}\langle -1,0,2\rangle=\left\langle\frac{7}{5},0,-\frac{14}{5}\right\rangle.$$

3. **(a)** We have $\overrightarrow{PQ} = \langle 1, 3, -2\rangle$, $\overrightarrow{QR} = \langle 4, -2, -1\rangle$, and $\overrightarrow{RP} = \langle -5, -1, 3\rangle$. We have

$$\begin{aligned} \overrightarrow{PQ}\cdot\overrightarrow{QR} &= 4-6+2=0 \\ \overrightarrow{QR}\cdot\overrightarrow{RP} &= -20+2-3=-21 \\ \overrightarrow{RP}\cdot\overrightarrow{PQ} &= -5-3-6=-14 \end{aligned}$$

We observe that the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are perpendicular, since their dot product is zero. So the angle $\angle PQR$ is a right angle, and the triangle ΔPQR is right.

(b) The perimeter of ΔPQR is

$$||\overrightarrow{PQ}|| + ||\overrightarrow{QR}|| + ||\overrightarrow{RP}|| = \sqrt{1+9+4} + \sqrt{16+4+1} + \sqrt{25+1+9} = \sqrt{14} + \sqrt{21} + \sqrt{35}.$$

(c) Since ΔPQR is a right triangle, we can use the good old-fashioned formula $A = \frac{1}{2}bh$, with the two shorter sides of the triangle as the "base" and "height":

$$A = \frac{1}{2}\sqrt{14}\cdot\sqrt{21} = \frac{1}{2}\sqrt{6\cdot 7^2} = \frac{7\sqrt{6}}{2}.$$

Even if we had not made this astute observation, we could have used the cross product (which works even when our vectors are not perpendicular). First we find $\overrightarrow{PQ}\times\overrightarrow{QR}$:

$$\begin{aligned} \overrightarrow{PQ}\times\overrightarrow{QR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 4 & -2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -2 \\ -2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix}\mathbf{k} \\ &= -7\mathbf{i} - 7\mathbf{j} - 14\mathbf{k} \\ &= \langle -7, -7, -14\rangle \end{aligned}$$

So we have

$$A = \frac{1}{2}||\overrightarrow{PQ}\times\overrightarrow{QR}|| = \frac{1}{2}\cdot\sqrt{7^2+7^2+4\cdot 7^2} = \frac{7\sqrt{6}}{2}.$$

4. **(a)** First we'll examine the behavior of this function (call it f) from the right; that is, on points of the form $(t, 0)$, where $t > 0$. We have

$$f(t,0) = \frac{t^3-0+0}{t^2+0} = t.$$

So $\lim_{t\to 0^+} f(t,0) = \lim_{t\to 0^+} t = 0$. Now let's look at the behavior of f from the upper-right; that is, on points of the form (t, t), where $t > 0$. We have

$$f(t,t) = \frac{t^3-t^2+t^3}{t^2+t^2} = \frac{2t^3-t^2}{2t^2} = t - \frac{1}{2}.$$

So $\lim_{t\to 0^+} f(t,t) = \lim_{t\to 0^+}(t-\frac{1}{2}) = -\frac{1}{2}$. These limits are different; therefore, the limit of f at the origin cannot exist.

(b) Again, let's start by examining the behavior of this function (call it g) from the right; that is, on points of the form $(t, 0)$, where $t > 0$. We have

$$g(t,0) = \frac{t\cos 0 - t}{t^2} = \frac{0}{t^2} = 0.$$

So again, $\lim_{t\to 0^+} g(t,0) = \lim_{t\to 0^+} 0 = 0$. Now let's look at the behavior of g along the curve consisting of points of the form $(\sin^2 t, t)$, which passes through the origin at $t = 0$. We have

$$g(\sin^2 t, t) = \frac{\sin^2 t\cos t - \sin^2 t}{\sin^4 t + \sin^4 t} = \frac{\cos t - 1}{2\sin^2 t}.$$

This time, it takes a little work to compute the limit of our function as t approaches zero. If we plug in $t = 0$, we get the indeterminate form $\frac{0}{0}$, so we use L'Hopital's Rule:

$$\begin{aligned}
\lim_{t\to 0} g(2\sin^2 t, t) &= \lim_{t\to 0} \frac{\cos t - 1}{2\sin^2 t} \\
&= \lim_{t\to 0} \frac{-\sin t}{4\sin t\cos t} \qquad \text{(by L'Hopital)} \\
&= \lim_{t\to 0} \frac{-1}{4\cos t} \\
&= -\frac{1}{4}.
\end{aligned}$$

Again, this differs from the limit we got before, so we conclude that the limit of g at the origin does not exist.

5. (a)

(b) The velocity function $\mathbf{v}(t)$ is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle \cos t, 1, -\sin t\rangle,$$

and the acceleration function $\mathbf{a}(t)$ is given by

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -\sin t, 0, -\cos t\rangle.$$

(c) The unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{\langle \cos t, 1, -\sin t\rangle}{\sqrt{\cos^2 t + 1 + \sin^2 t}} = \frac{1}{\sqrt{2}}\langle \cos t, 1, -\sin t\rangle = \left\langle \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}} \right\rangle.$$

The principal unit normal vector $\mathbf{N}(t)$ is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \frac{\left\langle -\frac{\sin t}{\sqrt{2}}, 0, -\frac{\cos t}{\sqrt{2}} \right\rangle}{\sqrt{\frac{\sin^2 t}{2} + \frac{\cos^2 t}{2}}} = \frac{\left\langle -\frac{\sin t}{\sqrt{2}}, 0, -\frac{\cos t}{\sqrt{2}} \right\rangle}{\frac{1}{\sqrt{2}}} = \langle -\sin t, 0, -\cos t\rangle.$$

(d) The length of the particle's path from $t = 0$ to $t = \pi$ is

$$\begin{aligned}
L &= \int_0^\pi ||\mathbf{r}'(t)||dt \\
&= \int_0^\pi \sqrt{2}dt \\
&= [\sqrt{2}t]_0^\pi \\
&= \sqrt{2}\pi.
\end{aligned}$$

6. **(a)** The normal vectors to the two given planes are $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3\rangle$, respectively. The two planes intersect in a line; to find the direction of this line, we must find a direction vector that is perpendicular to both normal vectors. This is a job for the cross product:

$$\begin{aligned}
\mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{k} \\
&= 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \\
&= \langle 5, -2, -3\rangle
\end{aligned}$$

So the direction vector of our line is $\langle 5, -2, -3\rangle$. To find the equation of our line, we must also have a point on the line; that is, a point on both planes. To find a point satisfying both of the equations $x + y + z = 1$ and $x - 2y + 3z = 1$, let's try taking $z = 0$ and solving for x and y. Setting $z = 0$ yields the equations $x + y = 1$ and $x - 2y = 1$. Subtracting these gives $3y = 0$; so $y = 0$, and it follows immediately that $x = 1$. So the point $(1, 0, 0)$ is on both lines. So our line is given by the equation

$$\frac{x-1}{5} = \frac{y-0}{-2} = \frac{z-0}{-3}; \qquad \text{that is,} \qquad \frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}.$$

(b) Let P be the point $(1, 0, 0)$, which lies on our line, and let Q be the point $(2, 0, 1)$. The vector $\overrightarrow{PQ}$ may not be perpendicular to the line, in which case we cannot conclude that the length of this vector is the distance from Q to the line. However, this is still a good place to start. So observe that $\overrightarrow{PQ} = \langle 1, 0, 1\rangle$, and the angle θ between this vector and the line is given by the equation

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \langle 5, -2, -3\rangle}{||\overrightarrow{PQ}|| \cdot ||\langle 5, -2, -3\rangle||} = \frac{2}{\sqrt{2} \cdot \sqrt{38}} = \frac{1}{\sqrt{19}}.$$

But then the length of a perpendicular segment from the point Q to the line is given by

$$\begin{aligned}
d &= ||\overrightarrow{PQ}|| \sin\theta \\
&= \sqrt{2} \cdot \sqrt{1 - \cos^2\theta} \\
&= \sqrt{2} \cdot \sqrt{1 - \frac{1}{19}} \\
&= \sqrt{2} \cdot \frac{\sqrt{18}}{\sqrt{19}} \\
&= \frac{6}{\sqrt{19}}.
\end{aligned}$$