Emerging Scholars Program – Fall 2007

M210E – Calculus Workshop

Practice Exam 2 – Silver Version

Instructions: This exam contains six problems. Problems 1 through 4 are worth 15 points each; problems 5 and 6 are worth 20 points each. You have 75 minutes to work on this test. No calculators, books, notes, or other external aids are allowed. The group responsible for each problem is indicated at the beginning of the problem; some problems have been edited and/or modified.

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1. (Hilbert) Define $f(x, y)$ by

$$f(x, y) = (x + 3y)e^{2x-y}.$$

(a) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) Use the derivatives of f at $(0, 0)$ to estimate the value of $f(0.1, -0.03)$.

2. (Fermat) Let $P = (3, 1, 0)$, $Q = (3, 0, -1)$, $R = (5, 1, 3)$, and $S = (0, -4, 0)$.

(a) Find the symmetric equations of the lines $\overleftrightarrow{QR}$ and $\overleftrightarrow{PS}$.

(b) Find the volume of the parallelepiped with vertex P and edges PQ, PR, and PS.

3. (Leibniz) Define a vector-valued function $\mathbf{r}(t)$ by

$$\mathbf{r}(t) = 2e^t\mathbf{i} + t\mathbf{j} - e^{2t}\mathbf{k}.$$

(a) Find the unit tangent vector to $\mathbf{r}(t)$.

(b) Find the length of the curve $\mathbf{r}(t)$ from $t = 0$ to $t = 2$.

4. (Gauss) Let $\mathcal{P}$ be the plane containing the point $(4, 1, 2)$ and the line $x = 4t$, $y = 3 + 2t$, $z = 1 + t$. Let $\mathcal{Q}$ be the plane containing the point $(4, 1, 2)$ and the line $x = 1 + 2t$, $y = 3 + t$, $z = 2t$.

(a) Find the standard-form equations of the planes $\mathcal{P}$ and $\mathcal{Q}$.

(b) What is the acute angle between the planes $\mathcal{P}$ and $\mathcal{Q}$? (If necessary, you may use inverse trig functions in your answer.)

5. (Hilbert) Consider the lines

$$\frac{x+3}{2} = \frac{y+5}{4} = \frac{z-5}{-1} \qquad \text{and} \qquad x = 1,\ y = z.$$

(a) Give the parametric equations for these two lines.

(b) Show that the two lines intersect, and find the acute angle between them.

(c) Find the symmetric equation of the line that is perpendicular to both of these lines.

6. (Hilbert) Define

$$f(x, y) = \frac{x^3 + y^3}{x^2 + y^2}.$$

In this problem, we will show that $\lim_{(x,y)\to(0,0)} f(x, y)$ exists.

(a) Using a path of your choice, determine the value of this limit.

(b) Using an epsilon-delta argument, prove that $\lim_{(x,y)\to(0,0)} f(x, y)$ exists and is equal to the value you gave in part (a). (*Hint:* At some point in your argument, you may find it useful to use the inequality $|a + b| = |a| + |b|$ for all real numbers a and b.)

Solutions

1. (a) By the Product Rule, we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}[x+3y]\cdot e^{2x-y} + (x+3y)\cdot\frac{\partial}{\partial x}\left[e^{2x-y}\right]\\ &= e^{2x-y} + 2(x+3y)e^{2x-y}\\ &= (2x+6y+1)e^{2x-y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}[x+3y]\cdot e^{2x-y} + (x+3y)\cdot\frac{\partial}{\partial y}\left[e^{2x-y}\right]\\ &= 3e^{2x-y} - (x+3y)e^{2x-y}\\ &= (-x-3y+3)e^{2x-y}\end{aligned}$$

(b) First we need to compute the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0,0)$. We have

$$f_x(0,0) = (0+0+1)e^0 = 1 \qquad \text{and} \qquad f_y(0,0) = 0+0+3)e^0 = 3.$$

So we approximate $f(0.1,-0.03)$ as follows:

$$\begin{aligned}f(0.1,-0.03) &\approx f(0,0) + 0.1\cdot f_x(0,0) - 0.03\cdot f_y(0,0)\\ &= 0 + 0.1 - 0.09\\ &= 0.01.\end{aligned}$$

2. (a) First we observe that $\overrightarrow{QR} = (3,0,-1)-(5,1,3) = \langle -2,-1,-4\rangle$. Using this and the fact that the point $(3,0,-1)$ lies on the line $\overleftrightarrow{QR}$, we obtain the following symmetric equation for $\overleftrightarrow{QR}$:

$$\frac{x-3}{-2} = \frac{y}{-1} = \frac{z+1}{-4}.$$

We can rewrite this by multiplying each quantity by -1:

$$\frac{x-3}{2} = \frac{y}{1} = \frac{z+1}{4}$$

Now observe that $\overrightarrow{PS} = (0,-4,0)-(3,1,0) = \langle 3,5,0\rangle$. The z-component of this direction vector is 0, which means that we must use two separate equations to represent the line $\overleftrightarrow{PS}$. Using the fact that $(0,-4,0)$ lies on this line, we obtain the following equations:

$$\frac{x}{3} = \frac{y+4}{5},\ z=0.$$

(Note that the equation $z=0$ comes from the fact that the z-coordinate of $(0,-4,0)$ is zero, *not* the fact that the z-component of the direction vector is zero.)

(b) We can find the volume of the given parallelepiped using the triple scalar product; see your textbook for an explanation of why this approach works. Our goal is to compute $\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})$. We'll start by computing $\overrightarrow{PR}\times\overrightarrow{PS}$:

$$\begin{aligned}\overrightarrow{PR}\times\overrightarrow{PS} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & 3\\ -3 & -5 & 0\end{vmatrix}\\ &= \begin{vmatrix}0 & 3\\ -5 & 0\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 3\\ -3 & 0\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & 0\\ -3 & -5\end{vmatrix}\mathbf{k}\\ &= 15\mathbf{i} - 9\mathbf{j} - 10\mathbf{k}\\ &= \langle 15,-9,-10\rangle\end{aligned}$$

So we have

$$\begin{aligned}\text{Volume} &= |\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})| \\ &= |\langle 0,1,-1\rangle\cdot\langle 15,-9,-10\rangle| \\ &= |0\cdot 15 + 1\cdot -9 + -1\cdot -10| \\ &= 1.\end{aligned}$$

3. (a) The unit tangent vector is given by

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \\ &= \frac{2e^t\mathbf{i}+\mathbf{j}-2e^{2t}\mathbf{k}}{\sqrt{(2e^t)^2+1^2+(-2e^{2t})^2}} \\ &= \frac{2e^t\mathbf{i}+\mathbf{j}-2e^{2t}\mathbf{k}}{\sqrt{1+4e^{2t}+4e^{4t}}} \\ &= \frac{2e^t\mathbf{i}+\mathbf{j}-2e^{2t}\mathbf{k}}{1+2e^{2t}} \\ &= \frac{2e^t}{1+2e^{2t}}\,\mathbf{i}+\frac{1}{1+2e^{2t}}\,\mathbf{j}-\frac{2e^{2t}}{1+2e^{2t}}\,\mathbf{k}\end{aligned}$$

(b) The length of the path described by $\mathbf{r}(t)$ from $t=0$ to $t=2$ is given by

$$\begin{aligned}L &= \int_0^2 ||\mathbf{r}'(t)||dt \\ &= \int_0^2 (1+2e^{2t})dt \qquad \text{(from above)} \\ &= [t+e^{2t}]_0^2 \\ &= (2+e^4)-(0+1) \\ &= 1+e^4.\end{aligned}$$

4. (a) To find the equation of $\mathcal{P}$, we need to find a point and two vectors that lie in $\mathcal{P}$. The point $(4,1,2)$ is given; and the given line has direction vector $\langle 4,2,1\rangle$. To obtain another vector in $\mathcal{P}$, we can use the fact that the point $(0,3,1)$ lies on the given line; so the vector $(4,1,2)-(0,3,1)=\langle 4,-2,1\rangle$ lies in the plane. We can now find the cross product of these vectors to obtain a normal vector to $\mathcal{P}$:

$$\begin{aligned}\overrightarrow{PR}\times\overrightarrow{PS} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 2 & 1\\ 4 & -2 & 1\end{vmatrix} \\ &= \begin{vmatrix}2 & 1\\ -2 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}4 & 1\\ 4 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}4 & 2\\ 4 & -2\end{vmatrix}\mathbf{k} \\ &= 4\mathbf{i}-16\mathbf{k} \\ &= \langle 4,0,-16\rangle\end{aligned}$$

So the vector $\langle 4,0,-16\rangle$ is normal to $\mathcal{P}$. Since $\mathcal{P}$ contains the point $(4,1,2)$, the equation of the plane is $4(x-4)-16(z-2)=0$, which simplifies to $4x-16z=-16$.

We must now perform the same computation for $\mathcal{Q}$. Again, the point $(4,1,2)$ is in our plane, as is the point $(1,3,0)$, which is on the given line. So the vector $(4,1,2)-(1,3,0)=\langle 3,-2,2\rangle$ lies in $\mathcal{Q}$, as does the direction vector of the given line, which is $\langle 2,1,2\rangle$. So we take the cross product of these two vectors to obtain a normal vector to $\mathcal{Q}$:

$$\begin{aligned}
\overrightarrow{PR}\times\overrightarrow{PS} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 2 \\ 2 & 1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{k} \\
&= -6\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \\
&= \langle -6,-2,7\rangle
\end{aligned}$$

So since $\mathcal{Q}$ contains the point $(4,1,2)$, the equation of $\mathcal{Q}$ is $-6(x-4)-2(y-1)+7(z-2)=0$, which simplifies to $-6x-2y+7z=-12$.

(b) To find the angle between $\mathcal{P}$ and $\mathcal{Q}$, we will first find the angle θ between their normal vectors (which we'll call $\mathbf{p}$ and $\mathbf{q}$). We know that

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{p}\cdot\mathbf{q}}{||\mathbf{p}||\cdot||\mathbf{q}||} \\
&= \frac{\langle 4,0,-16\rangle\cdot\langle -6,-2,7\rangle}{||\langle 4,0,-16\rangle||\cdot||\langle -6,-2,7\rangle||} \\
&= \frac{-24+0-112}{4\sqrt{17}\cdot\sqrt{89}} \\
&= \frac{-136}{4\sqrt{1513}} \\
&= \frac{-34}{\sqrt{1513}}.
\end{aligned}$$

So $\theta=\cos^{-1}\left(\frac{-34}{\sqrt{1513}}\right)$, which is an obtuse angle. We're looking for the acute angle between the two planes; this is the supplement of θ, which is $\cos^{-1}\left(\frac{34}{\sqrt{1513}}\right)$.

5. **(a)** The first line contains the point $(-3,-5,5)$ and has direction vector $\langle 2,4,-1\rangle$, so it is described by the parametric equations $x=-3+2t$, $y=-5+4t$, $z=5-t$. The second line contains the point $(1,0,0)$ and has direction vector $\langle 0,1,1\rangle$, so it is described by the parametric equations $x=1$, $y=t$, $z=t$.

(b) To show that the two lines intersect, we will try to find a point on the first line satisfying the equations $x=1$ and $y=z$, which describe the second line. Plugging $x=1$ and $z=y$ into the symmetric equation for the first line yields

$$\frac{1+3}{2}=\frac{y+5}{4}=\frac{y-5}{-1}.$$

After some rearrangement, this gives the equation $8=y+5=20-4y$. The value $y=3$ satisfies these equations; and using the fact that $y=z$, we get $z=3$. So the point $(1,3,3)$ lies on both lines. So the lines do intersect. To find the angle between them, we find the angle θ between their direction vectors, which we call **a** and **b**:

$$\begin{aligned}\cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||\cdot||\mathbf{b}||}\\ &= \frac{\langle 2,4,-1\rangle\cdot\langle 0,1,1\rangle}{||\langle 2,4,-1\rangle||\cdot||\langle 0,1,1\rangle||}\\ &= \frac{3}{\sqrt{21}\cdot\sqrt{2}}\\ &= \frac{3}{\sqrt{42}}.\end{aligned}$$

So the acute angle between the two lines is $\cos^{-1}\left(\frac{3}{\sqrt{42}}\right)$.

6. (a) We'll evaluate the limit of f from the right-hand side; that is, along points of the form $(t,0)$, where $t>0$. We have

$$\begin{aligned}\lim_{t\to 0^+} f(t,0) &= \lim_{t\to 0^+}\frac{t^3+0}{t^2+0}\\ &= \lim_{t\to 0^+} t\\ &= 0.\end{aligned}$$

So if the limit exists, it must be equal to 0.

(b) Now we will show that the limit in question is indeed equal to 0. Let $\epsilon>0$. Take $\delta=\epsilon$; we will show that, for all points (x,y) within a distance of δ away from $(0,0)$ (but not equal to $(0,0)$), we have $|f(x,y)-0|<\epsilon$.

So let (x,y) be a point within δ of $(0,0)$ but not equal to $(0,0)$. Then we know that $|x|<\epsilon$ and $|y|<\epsilon$. So we have

$$\begin{aligned}|x^3+y^3| &\le |x^3|+|y^3|\\ &= |x|\cdot x^2+|y|\cdot y^2\\ &< \epsilon\cdot x^2+\epsilon\cdot y^2\\ &= \epsilon(x^2+y^2).\end{aligned}$$

So we see that $|x^3+y^3|<\epsilon(x^2+y^2)$. From this, it follows that

$$\frac{|x^3+y^3|}{x^2+y^2}<\epsilon,$$

and thus

$$\left|\frac{x^3+y^3}{x^2+y^2}\right|<\epsilon.$$

So $|f(x,y)|<\epsilon$, as desired.