

Emerging Scholars Program – Fall 2007  
M210E – Calculus Workshop  
Practice Exam 3 – Gold Version

*Instructions: This exam contains ten problems. Each problem is worth 20 points. You have three hours to work on this test. No calculators, books, notes, or other external aids are allowed. The group responsible for each problem is indicated at the beginning of the problem; some problems have been edited and/or modified.*

**If you will only be taking the test for 90 minutes, please work questions 6 through 10 only.**



**1. (Euler, Hilbert)** Let's start off with a series question.

(a) Evaluate the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{n}$ . Show all of your work (do not assume any knowledge about this particular limit that you may remember from class).

(b) Prove or disprove that the following series converges:

$$\sum_{n=2}^{\infty} \frac{n}{(\log n)^n}$$

2. (Hilbert) Define a function  $f(x)$  by

$$f(x) = \frac{x}{1 + 2x^2 + x^4}.$$

(a) Find the Taylor series, centered at  $x = 0$ , for  $f(x)$ .

(b) Use the Taylor series obtained in (a) to approximate the value of  $f(0.1)$ , correct to within an error of  $\pm 0.0001$ . Show that the error of your approximation is within the desired margin of error.

3. (Pascal) Define  $P = (2, 0, 2)$ ,  $Q = (4, 8, 12)$ ,  $R = (6, -2, 4)$  and  $S = (12, 4, 16)$ .

(a) Compute the area of the triangle  $\Delta PQR$ .

(b) Determine whether the points  $P$ ,  $Q$ ,  $R$ , and  $S$  are coplanar.

4. **(Fourier)** Define  $P = (1, 2, 3)$ ,  $Q = (2, 2, 0)$ , and  $R = (3, -1, 5)$ .

(a) Find the parametric and symmetric equations of the line  $\overleftrightarrow{QR}$ .

(b) Find the equation of the plane containing the point  $P$  and the line  $\overleftrightarrow{QR}$ .

(c) Compute the distance from the point  $P$  to the line  $\overleftrightarrow{QR}$ .

5. **(Hilbert)** A particle in coordinate space is located at the origin at time  $t = 0$ ; the velocity vector of the particle at time  $t = 0$  is  $\langle 2, 1, 2 \rangle$ . The particle accelerates with a constant acceleration function  $\mathbf{a}(t) = \langle 1, 0, -1 \rangle$ .

(a) Find the velocity and position functions  $\mathbf{v}(t)$  and  $\mathbf{r}(t)$ .

(b) Find the unit tangent vector to the path of the particle at time  $t = 1$ .

(c) Let  $L$  be the length of the path travelled by the particle from  $t = 0$  to  $t = 1$ . Write  $L$  as an integral and simplify the integrand as much as possible (but don't worry about evaluating the integral).

6. **(Riemann)** Define  $f(x, y, z) = xe^{2yz}$  for all  $(x, y, z) \in \mathbb{R}$ .

(a) Compute  $\vec{\nabla} f(x, y, z)$ .

(b) Let  $\mathbf{v}$  be the vector  $\langle 2, -2, 1 \rangle$ . Compute the directional derivative of  $f$  at the point  $(3, 0, 2)$  in the direction of the vector  $\mathbf{v}$ .

(c) Observe that the point  $(3, 0, 2)$  lies on the surface  $xe^{2yz} = 3$ . Give the equation of the tangent plane to this surface at the point  $(3, 0, 2)$ .

7. **(Euler)** Define  $f(x, y, z) = x^3 + y^3 + z^3$  for all  $(x, y, z) \in \mathbb{R}^3$ .

(a) Given that  $x = 3st^2$ ,  $y = 3st^3$ , and  $z = 3st^4$  (where  $s$  and  $t$  are independent variables), find  $\frac{\partial f}{\partial s}$ . Write your answer in terms of the variables  $s$  and/or  $t$ .

(b) Find the maximum and minimum value of  $f$  on the region in  $\mathbb{R}^3$  defined by the inequality  $x^2 + y^2 + z^2 \leq 1$ .

8. **(Gauss)** Let  $X$  be the set of points in  $\mathbb{R}^3$  satisfying the equations  $xy = 1$  and  $y^2 + z^2 = 1$ .

(a) Each of the equations  $xy = 1$  and  $y^2 + z^2 = 1$ , taken individually, defines a surface. In separate pictures, draw these two surfaces.

(b) The set  $X$  consists of two curves, which are reflections of each other. Let  $Y$  be the curve that lies in the half-space  $x > 0$ . Let  $f(x, y, z) = xy + yz$  for all  $(x, y, z) \in \mathbb{R}^3$ . Find the maximum and minimum values of the function  $f$  on the curve  $Y$ , if these values exist. If one or both of these values do not exist, explain why.

9. (**Fermat**) Define  $R$  to be the region in  $\mathbb{R}^2$  consisting of all points  $(x, y)$  that satisfy the inequalities

$$1 \leq y \leq 3 \quad \text{and} \quad x \leq y \leq x^3.$$

(a) Sketch this region in the coordinate plane.

(b) Define  $f(x, y) = \frac{x^2}{y^2}$  for all ordered pairs  $(x, y)$  of real numbers such that  $y \neq 0$ . Compute the double integral

$$\iint_R f(x, y) \, dA.$$

10. (Hilbert) For all real numbers  $x$  and  $y$ , define

$$F(x, y) = \int_0^x \int_0^y ste^{-s^2t^2} dt ds.$$

It is a fact, which you may assume throughout this problem, that the function  $F$  is differentiable on  $\mathbb{R}^2$ .

(a) Show that  $F$  has a critical point at the point  $(0,0)$ . Is this critical point a local minimum, a local maximum, or a saddle point? (*Hint:* You should be able to answer this part without doing any differentiation or integration.)

(b) Let  $(x, y)$  be a point in the first quadrant of  $\mathbb{R}^2$ . Compute the partial derivatives  $\frac{\partial F}{\partial x}$  and  $\frac{\partial F}{\partial y}$  at the point  $(x, y)$ .

## Solutions

1. (a) Define  $L = \lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} n^{1/n}$ . Then we have

$$\begin{aligned}\log L &= \lim_{n \rightarrow \infty} \frac{1}{n} \log n \\ &= \lim_{n \rightarrow \infty} \frac{\log n}{n} \quad (\text{note that this is a limit of the form } \infty/\infty) \\ &= \lim_{n \rightarrow \infty} \frac{1/n}{1} \quad (\text{by L'Hopital's Rule}) \\ &= 0.\end{aligned}$$

So since  $\log L = 0$ , we have  $L = e^0 = 1$ .

(b) We will use the Root Test to determine whether the given series converges. Letting  $a_n = \frac{n}{(\log n)^n}$ , we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{(\log n)^n}} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{\log n} \\ &= \frac{1}{\infty} \\ &= 0.\end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 0 < 1$ , the series converges by the Root Test.

2. (a) First, observe that

$$f(x) = \frac{x}{1 + 2x^2 + x^4} = \frac{x}{(1 + x^2)^2}.$$

Notice that we can take the antiderivative of this function to obtain a function whose Taylor series we know. We have

$$\begin{aligned}f(x) &= \frac{d}{dx} \left[ -\frac{1}{2} \cdot \frac{1}{1 + x^2} \right] \\ &= \frac{d}{dx} \left[ -\frac{1}{2} \sum_{n=0}^{\infty} (-x^2)^n \right] \\ &= \frac{d}{dx} \left[ \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n} \right] \\ &= \sum_{n=0}^{\infty} \frac{2n \cdot (-1)^{n+1} x^{2n-1}}{2} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} n x^{2n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{2n-1} \quad \text{since the } n = 0 \text{ term of this series is zero.}\end{aligned}$$

(b) We know that

$$f(0.1) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n \cdot (0.1)^{2n-1}.$$

This is an alternating series; therefore, we can use the fact that if we sum the first  $k$  terms of this series, the error is no greater than the absolute value of the  $(k+1)^{th}$  term. Observe that the first few terms of this series are

$$f(0.1) = 1(0.1) - 2(0.1)^3 + 3(0.1)^5 - \dots$$

We have  $3(0.1)^5 = 0.00003 < 0.0001$ , so it suffices to add the first two terms of the series:

$$f(0.1) \approx 1(0.1) - 2(0.1)^3 = 0.1 - 0.002 = 0.098.$$

We know, by properties of alternating series, that the actual sum of the infinite series lies in the interval  $[S_2, S_3] = [0.098, 0.09803]$  (where  $S_k$  denotes the  $k^{th}$  partial sum of the series), so we are done.

3. (a) First, we find the vectors  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ :

$$\begin{aligned}\overrightarrow{PQ} &= (4, 8, 12) - (2, 0, 2) = \langle 2, 8, 10 \rangle \\ \overrightarrow{PR} &= (6, -2, 4) - (2, 0, 2) = \langle 4, -2, 2 \rangle\end{aligned}$$

We can now use the cross product to find the area of  $\triangle PQR$ . We know that

$$\text{Area} = \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\|,$$

so it remains to find the cross product of these two vectors:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 8 & 10 \\ 4 & -2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 8 & 10 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 10 \\ 4 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 8 \\ 4 & -2 \end{vmatrix} \mathbf{k} \\ &= 36\mathbf{i} + 36\mathbf{j} - 36\mathbf{k} \\ &= \langle 36, 36, -36 \rangle\end{aligned}$$

So the area of  $\triangle PQR$  is

$$\text{Area} = \frac{1}{2} \|\langle 36, 36, -36 \rangle\| = \frac{1}{2} \sqrt{3 \cdot 36^2} = \frac{36\sqrt{3}}{2} = 18\sqrt{3}.$$

(b) First note that  $\overrightarrow{PS} = (12, 4, 16) - (2, 0, 2) = \langle 10, 4, 14 \rangle$ . We know that the points  $P$ ,  $Q$ ,  $R$ , and  $S$  are coplanar if and only if the vectors  $\overrightarrow{PQ}$ ,  $\overrightarrow{PR}$ , and  $\overrightarrow{PS}$  are coplanar; that is, if and only if the volume of the parallelepiped defined by these three vectors is zero. Recall that the volume of the parallelepiped defined by three vectors is equal to the absolute value of the triple scalar product of the vectors. So we need to determine whether the triple scalar product  $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR})$  is zero. (Note that we can put the three vectors in whichever order we want; it is advantageous to choose this order since we already have the cross product of  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ .) We have

$$\begin{aligned}\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) &= \langle 10, 4, 14 \rangle \cdot \langle 36, 36, -36 \rangle \\ &= 10 \cdot 36 + 4 \cdot 36 + 14 \cdot -36 \\ &= (10 + 4 - 14) \cdot 36 \\ &= 0.\end{aligned}$$

So the volume of the parallelepiped defined by the three vectors is zero; and so the points  $P$ ,  $Q$ ,  $R$ , and  $S$  are indeed coplanar.

4. (a) The direction vector of the line in question is

$$\overrightarrow{QR} = (3, -1, 5) - (2, 2, 0) = \langle 1, -3, 5 \rangle.$$

The point  $Q = (2, 2, 0)$  lies on the line, so we can use this information to write the parametric equations for  $\overrightarrow{QR}$ . We know that this line consists of all points of the form  $(2, 2, 0) + t\langle 1, -3, 5 \rangle$  where  $t$  is a real number; therefore, the parametric equations are

$$x = 2 + t \quad y = 2 - 3t \quad z = 5t.$$

The direction vector does not have any components equal to zero, so we can use the usual template for the symmetric equation of the line:

$$\frac{x-2}{1} = \frac{y-2}{-3} = \frac{z-0}{5}.$$

(b) We know that the direction vector  $\langle 1, -3, 5 \rangle$  lies in this plane, as does the point  $(2, 2, 0)$ . So it remains only to find one more direction vector that lies in the plane. For this, we use the vector  $\overrightarrow{PQ}$ :

$$\overrightarrow{PQ} = (2, 2, 0) - (1, 2, 3) = \langle 1, 0, -3 \rangle$$

Now we can use the two direction vectors to find a vector normal to the desired plane:

$$\begin{aligned} \langle 1, -3, 5 \rangle \times \langle 1, 0, -3 \rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 5 \\ 1 & 0 & -3 \end{vmatrix} \\ &= \begin{vmatrix} -3 & 5 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 5 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -3 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\ &= 9\mathbf{i} + 8\mathbf{j} + 3\mathbf{k} \\ &= \langle 9, 8, 3 \rangle \end{aligned}$$

We can now use this normal vector and the point  $(2, 2, 0)$  to write the equation of the plane:

$$9(x-2) + 8(y-2) + 3(z-0) = 0$$

This simplifies to

$$9x + 8y + 3z = 34.$$

(c) To find the distance from the point  $(1, 2, 3)$  to the line  $\frac{x-2}{1} = \frac{y-2}{-3} = \frac{z}{5}$ , we will use the vector  $\overrightarrow{PQ} = \langle 1, 0, -3 \rangle$ , which runs from the point  $P$  to the line  $\overrightarrow{QR}$ , and the vector  $\overrightarrow{QR} = \langle 1, -3, 5 \rangle$ , which runs along the line. We have

$$\begin{aligned} \text{Distance} &= \frac{\|\overrightarrow{PQ} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|} \\ &= \frac{\|\langle -9, -8, -3 \rangle\|}{\|\langle 1, -3, 5 \rangle\|} \\ &= \frac{\sqrt{(-9)^2 + (-8)^2 + (-3)^2}}{\sqrt{1^2 + (-3)^2 + 5^2}} \\ &= \frac{\sqrt{154}}{\sqrt{35}} \\ &= \sqrt{\frac{22}{5}}. \end{aligned}$$

(Note that  $\overrightarrow{PQ} \times \overrightarrow{QR} = -(\overrightarrow{QR} \times \overrightarrow{PQ})$ ; the vector in parentheses on the right is the cross product that we calculated in part (b).)

5. (a) The velocity function  $\mathbf{v}(t)$  is given by

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(s) ds \\ &= \langle 2, 1, 2 \rangle + \int_0^t \langle 1, 0, -1 \rangle ds \\ &= \langle 2, 1, 2 \rangle + \langle t, 0, -t \rangle \\ &= \langle 2+t, 1, 2-t \rangle.\end{aligned}$$

The position function  $\mathbf{r}(t)$ , then, is given by

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(s) ds \\ &= \langle 0, 0, 0 \rangle + \int_0^t \langle 2+s, 1, 2-s \rangle ds \\ &= \langle 0, 0, 0 \rangle + \left( \left\langle 2t + \frac{t^2}{2}, t, 2t - \frac{t^2}{2} \right\rangle - \langle 0, 0, 0 \rangle \right) \\ &= \left\langle 2t + \frac{t^2}{2}, t, 2t - \frac{t^2}{2} \right\rangle\end{aligned}$$

(b) The unit tangent vector  $\mathbf{T}(t)$  is given by

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\ &= \frac{\langle 2+t, 1, 2-t \rangle}{\sqrt{(2+t)^2 + 1^2 + (2-t)^2}}.\end{aligned}$$

Evaluating this at  $t = 1$ , we get  $\mathbf{T}(1) = \frac{\langle 3, 1, 1 \rangle}{\sqrt{3^2 + 1^2 + 1^2}} = \left\langle \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right\rangle$ .

(c) The distance travelled by the particle from  $t = 0$  to  $t = 1$  is given by

$$\begin{aligned}L &= \int_0^1 \|\mathbf{r}'(t)\| dt \\ &= \int_0^1 \sqrt{(2+t)^2 + 1^2 + (2-t)^2} dt \\ &= \int_0^1 \sqrt{2t^2 + 9} dt.\end{aligned}$$

6. (a) We have

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz} \rangle.\end{aligned}$$

(b) In order to compute the directional derivative of  $f$  in the direction  $\mathbf{v}$ , we must first find a unit vector  $\mathbf{u}$  that is in the same direction as  $\mathbf{v}$ :

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, -2, 1 \rangle}{\sqrt{2^2 + (-2)^2 + 1^2}} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

So the directional derivative is

$$\begin{aligned} D_{\mathbf{u}}f(3, 0, 2) &= \overrightarrow{\nabla}f(3, 0, 2) \cdot \mathbf{u} \\ &= \langle e^0, 12e^0, 0 \rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle \\ &= 1 \cdot \frac{2}{3} + 12 \cdot -\frac{2}{3} + 0 \cdot \frac{1}{3} \\ &= -\frac{22}{3}. \end{aligned}$$

(c) We know that the gradient vector  $\overrightarrow{\nabla}f(3, 0, 2)$  gives us a normal vector to the surface defined by the equation  $f(x, y, z) = 3$  at the point  $(3, 0, 2)$ . The gradient at the point  $(3, 0, 2)$  is equal to  $\langle 1, 12, 0 \rangle$ ; therefore, the tangent plane to the surface at this point is normal to this vector and contains the point  $(3, 0, 2)$ . So the equation of this plane is

$$1(x - 3) + 12(y - 0) + 0(z - 2) = 0,$$

which simplifies to  $x + 12y = 3$ .

7. (a) By the Chain Rule, we have

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s} \\ &= 3x^2 \cdot 3t^2 + 3y^2 \cdot 3t^3 + 3z^2 \cdot 3t^4 \\ &= 3(3st^2)^2 \cdot 3t^2 + 3(3st^3)^2 \cdot 3t^3 + 3(3st^4)^2 \cdot 3t^4 \\ &= 81s^2t^6 + 81s^2t^9 + 81s^2t^{12} \\ &= 81s^2t^6(1 + t^3 + t^6). \end{aligned}$$

(b) Since the given region is closed and bounded, we know that the function  $f$  will attain maximum and minimum values on this region. We will then begin by looking for critical points of  $f$  in the interior of the region; that is, critical points satisfying  $x^2 + y^2 + z^2 < 1$ .

We have  $\overrightarrow{\nabla}f(x, y, z) = \langle 3x^2, 3y^2, 3z^2 \rangle$ . A point  $(x, y, z)$  is a critical point if and only if  $\overrightarrow{\nabla}f(x, y, z) = \langle 0, 0, 0 \rangle$ ; in this case, this occurs only at the point  $(0, 0, 0)$ . This point does satisfy the inequality  $x^2 + y^2 + z^2 < 1$ , so we will include this point in the list of points we check when we look for the maximum and minimum values of  $f$  on the region.

Now we must check for critical points on the boundary of our region; that is, on the sphere  $x^2 + y^2 + z^2 = 1$ . For this, we will use the method of Lagrange multipliers. Define  $g(x, y, z) = x^2 + y^2 + z^2 - 1$ , so that the surface in question is given by the equation  $g(x, y, z) = 0$ . We are then looking for points  $(x, y, z)$  such that  $\overrightarrow{\nabla}f(x, y, z) = \lambda \overrightarrow{\nabla}g(x, y, z)$  for some constant  $\lambda$ . That is, we want to find points  $(x, y, z)$  such that

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad \langle 3x^2, 3y^2, 3z^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle.$$

This yields the four equations

$$\begin{aligned}3x^2 &= 2\lambda x \\3y^2 &= 2\lambda y \\3z^2 &= 2\lambda z \\x^2 + y^2 + z^2 &= 1.\end{aligned}$$

Rearranging and factoring the first three yields  $x(3x - 2\lambda) = y(3y - 2\lambda) = z(3z - 2\lambda) = 0$ . In order for  $x(3x - 2\lambda)$  to be equal to zero, we must have either  $x = 0$  or  $x = \frac{2}{3}\lambda$ . Similarly, we must have  $y = 0$  or  $y = \frac{2}{3}\lambda$ ; and we must have  $z = 0$  or  $z = \frac{2}{3}\lambda$ . This analysis leads to  $2 \cdot 2 \cdot 2 = 8$  different cases; we organize them as follows.

**Case 1.** If all three of  $x$ ,  $y$ , and  $z$  are zero, then the equation  $x^2 + y^2 + z^2 = 1$  is not satisfied. So there are no points we need to check in this case.

**Case 2.** If exactly two of  $x$ ,  $y$ , and  $z$  are zero (let's say  $x$  and  $y$ ), then we must have  $z = \pm 1$  in order for the equation  $x^2 + y^2 + z^2 = 1$  to be satisfied. The other three equations will be satisfied as long as we take  $\lambda = \pm \frac{3}{2}$ , so we can add the points  $(0, 0, \pm 1)$  to our list. If we choose different pairs of coordinates to set equal to zero, we will get the points  $(\pm 1, 0, 0)$  and  $(0, \pm 1, 0)$ , for a total of six points in this case.

**Case 3.** If exactly one of  $x$ ,  $y$ , and  $z$  is zero (let's say  $x$ ), then we must have  $y = \frac{2}{3}\lambda = z$ . So  $y = z$ ; and since we have  $x^2 + y^2 + z^2 = 1$ , we must have  $y = z = \frac{1}{\sqrt{2}}$  or  $y = z = -\frac{1}{\sqrt{2}}$ . This yields the points  $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$  and  $(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ . By changing the coordinate that we set equal to zero, we get four more points, for a total of six in this case.

**Case 4.** If none of the coordinates are equal to zero, then we must have  $x = y = z = \frac{2}{3}\lambda$ . So  $x = y = z$ ; and since  $x^2 + y^2 + z^2 = 1$ , we have  $x = y = z = \frac{1}{\sqrt{3}}$  or  $x = y = z = -\frac{1}{\sqrt{3}}$ . This yields the points  $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$  and  $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ .

Now we have fifteen "candidate" points (one in the interior and  $6 + 6 + 2 = 14$  on the boundary) at which the maximum and minimum values of  $f$  may occur. We have  $f(0, 0, 0) = 0$ . Evaluating  $f$  at the points from Case 2 yields the values 1 and  $-1$ . Evaluating  $f$  at the points from Case 3 yields the values  $\pm \frac{1}{\sqrt{2}}$ ; and evaluating at the points from Case 4 yields the values  $\pm \frac{1}{\sqrt{3}}$ . Of these, the largest value is 1 and the smallest value is  $-1$ . So the maximum value of  $f$  on the given region is 1, and the minimum is  $-1$ .

8. (a) I won't bother drawing the surfaces here, but the first one can be gotten by drawing the hyperbola  $xy = 1$  in the plane  $z = 0$  and then sweeping this hyperbola up and down parallel to the  $z$ -axis; and the second one is an infinite cylinder of radius one centered around the  $x$ -axis.

(b) First let us consider the question of  $f$  actually has maximum and minimum values on  $Y$ . First, note that for all points  $(x, y, z)$  on  $Y$ , we have  $xy = 1$ ; so for points on  $Y$ , we have  $f(x, y, z) = 1 + yz$ . Even though the curve  $Y$  is not bounded, we can think about the "end behavior" of the function  $f$  as we move out along the curve. Both "ends" of the curve stretch out in the  $+x$ -direction; that is, as we move out along the curve, the  $x$ -coordinate goes to infinity and the  $y$ -coordinate goes to zero (from the positive side). Since the  $y$ -coordinate goes to zero, the  $z$ -coordinate goes to one. So at both "ends" of the curve, the value of  $f(x, y, z) = 1 + yz$  converges to  $1 + 0 \cdot 1 = 1$ .

Since the expression  $yz$  takes both positive and negative values on  $Y$  (since  $y$  is always positive but  $z$  is allowed to be positive or negative independently of  $y$ ), we can conclude that this function will have both a maximum and a minimum on  $Y$ . To find these, we will now use differential calculus.

Define  $g(x, y, z) = xy - 1$  and  $h(x, y, z) = y^2 + z^2 - 1$ , so that the curve  $Y$  is determined by the constraints  $g(x, y, z) = 0$  and  $h(x, y, z) = 0$  (along with the inequality  $x > 0$ ). It is a fact that, if  $f$  attains a maximum or a minimum at a point  $(x, y, z)$  on  $Y$ , then the gradient  $\vec{\nabla}f(x, y, z)$  at that point must be in the plane of  $\vec{\nabla}g(x, y, z)$  and  $\vec{\nabla}h(x, y, z)$ . That is, we must have

$$\langle y, x + z, y \rangle = \lambda \langle y, x, 0 \rangle + \mu \langle 0, 2y, 2z \rangle$$

for some constants  $\lambda$  and  $\mu$ . That is, we must have

$$\begin{aligned} y &= \lambda y \\ x + z &= \lambda x + 2\mu y \\ y &= 2\mu z \end{aligned}$$

Since  $y = \lambda y$  and  $y$  does not equal zero for any point on  $Y$ , we can conclude that  $\lambda = 1$ . Using this in the second equation yields  $z = 2\mu y$ . Substituting this into the third equation yields  $y = 4\mu^2 y$ . Again, we know that  $y \neq 0$ , so we can conclude that  $4\mu^2 = 1$ ; that is, that  $\mu = \frac{1}{2}$  or  $\mu = -\frac{1}{2}$ .

Therefore, we have  $y = z$  or  $y = -z$ . In the first case, using the fact that  $y^2 + z^2 = 1$  and  $y > 0$ , we find that  $y = z = \frac{1}{\sqrt{2}}$ ; in the latter case, we get  $y = \frac{1}{\sqrt{2}}$  and  $z = -\frac{1}{\sqrt{2}}$ . Since in each case we have  $x = \frac{1}{y}$ , this gives us two critical points:  $(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$  and  $(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ . The value of  $f$  at the first point is  $\frac{3}{2}$ , and the value of  $f$  at the second point is  $\frac{1}{2}$ . These are the maximum and minimum values of  $f$  on  $Y$ , respectively.

9. (a) Again, no sketch for you. Use a graphing calculator if you need confirmation that your picture is correct.

(b) Although it looks like the given region is defined as a Type I region, this is not exactly the case, since for some values of  $x$  the upper bound  $y = 3$  is the stricter of the two, and for others the upper bound  $y = x^3$  is the stricter of the two. If we want to solve this problem using only one integral (well, one double integral), we'll need to reformulate the given region as a Type II region. If your sketch is good (and you're good at inverting functions), you should be able to see that

$$R = \{(x, y) : 1 \leq y \leq 3 \text{ and } \sqrt[3]{y} \leq x \leq y\}.$$

So we have

$$\begin{aligned} \iint_R f(x, y) dA &= \int_1^3 \int_{\sqrt[3]{y}}^y \frac{x^2}{y^2} dx dy \\ &= \int_1^3 \left[ \frac{x^3}{3y^2} \right]_{\sqrt[3]{y}}^y dy \\ &= \int_1^3 \left( \frac{y^3}{3y^2} - \frac{y}{3y^2} \right) dy \\ &= \int_1^3 \left( \frac{y}{3} - \frac{1}{3y} \right) dy \\ &= \left[ \frac{y^2}{6} - \frac{\log y}{3} \right]_1^3 \\ &= \left( \frac{9}{6} - \frac{\log 3}{3} \right) - \left( \frac{1}{6} - 0 \right) \\ &= \frac{4}{3} - \frac{\log 3}{3}. \end{aligned}$$

10. (a) Observe that  $F(x, y) = 0$  for any point  $(x, y)$  that is on the  $x$ -axis or the  $y$ -axis (since the integral of any function from zero to zero is zero). So  $F_x(0, 0) = F_y(0, 0) = 0$ , since  $F$  is constant along both of the coordinate axes.

This point is, in fact, a local minimum of  $F$ . We will show this by showing that  $F(x, y) \geq 0$  for all  $(x, y) \in \mathbb{R}^2$ . We have already established this when  $x = 0$  or  $y = 0$ , so now assume that both coordinates are nonzero.

When  $x$  and  $y$  are both positive, the inner integral is the integral of a positive function from bottom to top, which is positive. Thus the outer integral is the integral of a positive function from left to right, which is positive.

When  $x$  is negative and  $y$  is positive, the inner integral is the integral of a negative function from bottom to top, which is negative. Thus the outer integral is the integral of a negative function from right to left, which is positive.

When  $x$  and  $y$  are both negative, the inner integral is the integral of a positive function from top to bottom, which is negative. Thus the outer integral is the integral of a negative function from right to left, which is positive.

When  $x$  is positive and  $y$  is negative, the inner integral is the integral of a negative function from top to bottom, which is positive. Thus the outer integral is the integral of a positive function from left to right, which is positive.

Therefore, the minimum value of  $F$  is zero; this minimum occurs (among other places) at the origin. Even though the origin is not the only place where the minimum value is achieved, it is still considered a local minimum by definition.

- (b) For convenience, define  $f(x, y) = xye^{-x^2y^2}$ . Let  $(x, y) \in \mathbb{R}^2$  such that  $x > 0$  and  $y > 0$ . By definition, we have

$$\begin{aligned} F_x(x, y) &= \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\int_0^{x+\Delta x} \int_0^y f(s, t) dt ds - \int_0^x \int_0^y f(s, t) dt ds}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\int_x^{x+\Delta x} \int_0^y f(s, t) dt ds}{\Delta x}. \end{aligned}$$

Since the function  $f$  is continuous, for small values of  $\Delta x$  we have

$$\int_x^{x+\Delta x} \int_0^y f(s, t) dt ds \approx \Delta x \cdot \int_0^y f(x, t) dt.$$

So since  $x \neq 0$ , the above limit reduces to

$$\begin{aligned} \int_0^y f(x, t) dt &= \int_0^y xte^{-x^2t^2} dt \\ &= \left[ -\frac{e^{-x^2t^2}}{2x} \right]_0^y \\ &= -\frac{e^{-x^2y^2}}{2x} + \frac{e^0}{2x} \\ &= \frac{1 - e^{-x^2y^2}}{2x}. \end{aligned}$$

So we have

$$F_x(x, y) = \frac{1 - e^{-x^2 y^2}}{2x}.$$

As for  $F_y(x, y)$ , observe that

$$\begin{aligned} F_y(x, y) &= \lim_{\Delta y \rightarrow 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\int_0^x \int_0^{y+\Delta y} f(s, t) dt ds - \int_0^x \int_0^y f(s, t) dt ds}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\int_0^{y+\Delta y} \int_0^x f(s, t) ds dt - \int_0^y \int_0^x f(s, t) ds dt}{\Delta y}, \end{aligned}$$

by Fubini's Theorem. From here, we can follow a parallel line of reasoning to see that

$$F_y(x, y) = \frac{1 - e^{-x^2 y^2}}{2y}.$$