Emerging Scholars Program – Fall 2007
M210E – Calculus Workshop
Problem Set 4

"It is the first duty of a hypothesis to be intelligible." – Thomas Henry Huxley

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19. (Really) Understanding convergence. Give a precise definition of what it means to say that a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a number L. Your definition should be devoid of subjective phrases such as "is close to" (how close?) and "gets large" (how large?). See if you can come up with a definition without referring to your notes or textbook. Then, use this definition to prove that the sequence $1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$ that you encountered in your homework does not converge to any real number.

20. 'Till Gabriel blows his horn. *Gabriel's horn* is a surface that can be constructed by taking the graph of the function $f(x) = \frac{1}{x}$ (for $x \geq 1$), and revolving it around the x-axis. Prove that Gabriel's horn has infinite surface area, but finite volume. (Thanks to the students in both sections who introduced this example to me. The first ESP problem generated by user feedback!)

21. Convergent or divergent? For each of the following sequences, determine, with proof, whether the given sequence converges or diverges.

$$a_n = \frac{5^n}{2^n + 3^n} \qquad a_n = \frac{5^n}{2^n \cdot 3^n} \qquad a_n = \tan^{-1}\left(\frac{n!}{n^2}\right) \qquad a_n = \left(1 + \frac{1}{n!}\right)^n$$

22. How fast does Fibonacci grow? The *Fibonacci sequence* $\{F_n\}_{n=0}^{\infty}$ is defined as follows. We define $F_0 = F_1 = 1$; and for $n \geq 2$, we define $F_n = F_{n-1} + F_{n-2}$. Thus the first few terms of the Fibonacci sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Compute the limit $\lim_{n\to\infty} F_{n+1}/F_n$. (You may find it useful to assume that this ratio has a limit; call it L. Don't worry right now about proving that this limit exists.) What does this limit tell you about the Fibonacci sequence?

23. A race to infinity. Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_n = \frac{(2n-1)!}{n^{2n-1}}$$

converges. Can you find its limit?

24. Ingenuity: Halve your cake and eat it too. A square cake is covered on the top and all sides with a uniform layer of icing.

1. Show some different ways in which one can cut this cake into two pieces, each of which has the same amount of cake and the same amount of icing.
2. Is it possible to cut the cake into three pieces so that each piece has the same amount of cake and the same amount of icing? How about five pieces? How about n pieces?